Classical Mechanics

Hyoungsoon Choi

Spring, 2014

Contents

Disclaimer: The world view represented in this section only holds for Newtonian physics and Galileian relativity. However, it is still very useful in developing physical intuitions relevant to classical mechanics. Basically, I will be treating you like a person born pre 1900, before quantum mechanics and special relativity.

Chapter 1

Introduction

Unlike quantum mechanics, in which our intuitive world view breaks down completely from the get go, there is no big secret in classical mechanics. The objects you are interested in are mostly visible and they respond to a *push* or *pull*, technically known as the force. The question here is "how do the objects move under given set of forces?" At the end of the day, your answer will be given as the position of an object as a function of time.

To achieve this goal, everything you learn in classical mechanics boils down to understanding the consequences of a single equation, $\mathbf{F} = m\mathbf{a}$. We will not pretend that this is a completely new concept you are yet to learn. Instead, we will deal with it exactly as what it is, something you have learned already, of which you do not understand the full consequences yet. In other words, throughout this course, we will learn how to interpret the equation and its consequences more carefully through various examples.

But before we seriously delve into mechanics, let's take a brief trip to France. In Sèvres, France, there is an underground vault that holds big chunks of metal that are made out of 90% platinum and10% iridium. There are three keys to this vault and you need all three to get in. Each year three people holding the keys to this vault gather together, go down to the vault open it up and make sure these metal chunks are still there unaltered. This seemingly bizarre behavior that can be mistaken as a ritual by platinum-iridium worshipping cult is actually of utmost importance to us that study science, especially physics.

The vault actually belongs to the International Bureau for Weights and Measures (Bureau International des Poids et Mesures, BIPM), and those metal pieces are the international prototype kilogram (IPK) and the international prototype meter (IPM). The sole purpose of BIPM's existence is to define a kilometer, a meter (and a second) and IPK and IPM are exactly that. There isn't any international prototype second (IPS) in that vault only because there is simply no physical object that can represent time. Now, what does this have anything to do with what we are about to study? As we will soon see, more than what you can imagine.

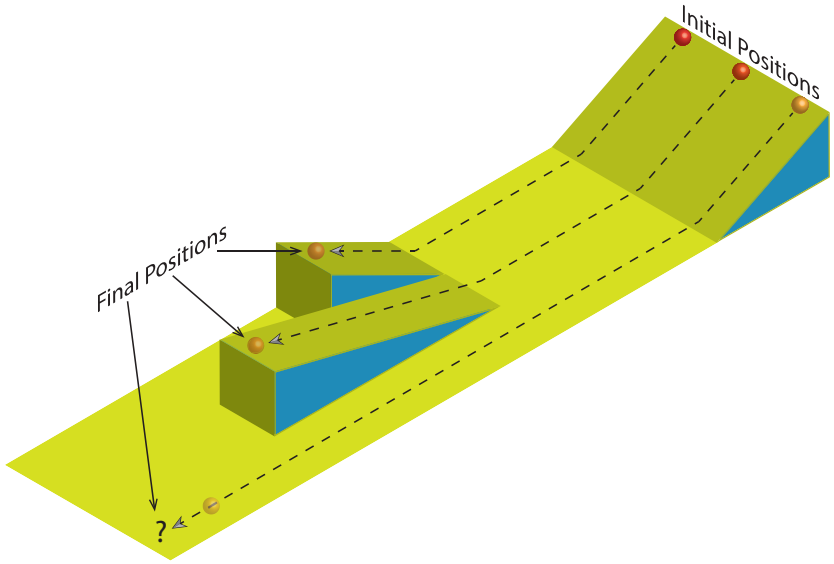

Figure 1.1: Galileo's thought experiment on inertia.

1.1 Kinematics and Kinetics

It should be noted that what we learn is usually dubbed as mechanics which encompasses kinetics and kinematics. Kinematics is an older field of study, in the sense, that a motion of a particle, a body, or a collection of them are studied without regard to why it happens, whereas kinetics deals with the motion in relation to its causes, *i.e.* forces and torques.

For example, a particle shot up at an angle under a constant vertical downward acceleration follows a parabolic trajectory. In kinematics, this is the only relevant information. Position, time and their relations are all there is to it. Although it is clear from Galileo's thought experiment of a body rolling down a slope that he understood the concept of inertia, his contribution to physics leaned more on kinematics than mechanics.

Kinetics is distinguished from kinematics by recognizing that the constant acceleration is a result of constant force acting on the object, and also the magnitude of acceleration depends on the magnitude of the force acting on it. This relation was established by Newton through his Laws of Motion, and only then did mechanics truly form by integrating kinematics and kinetics. Mechanics is after all understanding the implications of Newton's Second Law of Motion, $\mathbf{F} = \mathbf{ma}$. This adds extra dimension to mechanics that lacked in kinematics, and that last missing dimension is mass.

We now have three physical quantities, mass, distance and time to fully grasp the concept of classical mechanics, the very starting point of almost every scientific endeavour, and it is not by chance that the standard units, also known as SI units, have three elements: meter, kilogram and second.

Bear in mind that vision is, by far, the most far reaching sensory system among the human's five primary senses, and kinematics is a visual interpretation of motion and kinetics a more abstract interpretation. In other words, the position of an object is visible where as force or inertia or mass is not. Noting

where the Sun, the Moon and stars each day was a relatively simple task. Since time is, as we shall soon see, a measure of change, any change in position that we observe already includes the notion of time. In other words, observation of positional change is the kinematics itself. When we are observing objects, say stars in the sky, move, we are probing into its kinematics already. Why they were there when they were there was a lot more tricky business. And exactly for this reason, we start with kinematics.

1.2 Kinematics: Watching *Wallace and Gromit*

Figure 1.2: Wallace and Gromit.

Wallace and Gromit is one of the most beloved stop motion claymations, created by Nick Park in 1989. In a claymation, a malleable material, such as plasticine clay, is formed into desired shapes, forming characters and backgrounds and a still shot is taken. Then these characters and backgrounds are deformed to represent changes and movements, and another still shot is taken. By repeating this process thousands of times, kinematics of clay objects are,

well, *fabricated*. Nonetheless, it does allow us to peek into the key concepts in kinematics.

Suppose, the plasticine clay is formed into shapes and you take a still shot. After a few hours, you take another still shot, another after a few hours, and another and another and so on. The clay won't move itself and when you play a movie out of these still shots, you will end up with a very boring *movie*. No one would notice, if you pause the video during the showing of that movie, that is of course, if there is anyone coming to see this. Just by staring at this, no one would be able to tell what happens before what. In that little claymation universe, the time doesn't exist.

Time takes any meaning only if there are changes; something has to move. Let's take a look at two images in Fig. 1.2. We can tell by looking at them that they are not the same. Then how far did Wallace's cup move? By about the width of the cup. But how far is it? Is it big or small? If you try to answer this question, what you learn is that there is no absolute scale of length in physics. When you say the cup moved by the width of the cup, you are setting the size of the cup as the reference scale. Once you decide to use the size of the cup as your unit of length measurements, you can now say something has moved by three cups or eleven cups, etc.

However, spatial change alone is not sufficient to describe motion. The fact that the position of some objects changed self-creates this other dimension, time. Unless you are willing to accept the notion that an object can be at two different places at the same time, which you can't in classical physics, the fact that there is a change in position allows you to conclude only that these two images represent two different points in time. But, because you can never figure out how far apart in time these images are by simply looking at them, the concept of time exists, but not in truly meaningful way. Just like length, there is no absolute measure of time. To quantify, how much time has elapsed, you need to be able to compare changes.

Imagine an independently moving object, say a red ball moving from point A to point B. Even if we cannot tell in any absolute sense how long it took for the cup to move, we can say whether it took longer than the red ball to move from point A to point B. However, the point of reference vanishes as soon as the red ball stops moving. For this reason, it is better to pick a repetitive motion as a reference of time, such as the Earth's rotation or orbital motion around the Sun. We can then use one cycle of this motion as a unit of time and describe other objects' motion based on it. For example, we can say that Wallace's cup has moved by one and a half cup in 1/864000 of the Earth's rotation whereas Gromit's cup has moved by 0.8 cup in the same period of time, thereby Wallace's cup moved at a higher velocity.

This is precisely why we need IPK and IPS. Only after having a reference point for distance and time, such as the width of the cup and the duration of the Earth's one full rotation, can we define what motions are quantitatively and kinematics starts to make sense. IPM and IPS (if there was one) are the internationally agreed upon basis for this relativeness, and that is why it is so important to have a precise, easily reproducible definition of them and keep them unaltered. Currently, one meter is defined as *the length of the path travelled by light in vacuum during a time interval of 1/299,792,458 of a second.* One second is defined as *the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground*

state of the caesium 133 atom. We will learn how IPK is related to mechanics, shortly, but at this point, it is not too big a stretch to say that our understanding of mechanics is kept in that vault in France.

Now that we have established the concept of time and space, motion has a meaning. We can define velocity and acceleration, as the amount change in position in a given time and the amount of change of that change in a given time. The agreement on that time frame is what allows you to create claymations. In real life, it can take a really long time to create two successive shots. But because you set the time difference between any two successive shots equal when you are playing it in video, you can create a controlled motion, and suddenly the movie gets a life and becomes a form of entertainment, and also a subject of kinematics.

1.3 Inertia and Inertial Frame

Unlike length and time, mass is not a visually identifiable quantity. So the question follows: how do we identify mass? But this question is in fact, missing a more important point entirely: why do we even care about mass? Kinematics, such as the famous Kepler's Laws of planetary motion, can be sufficiently described by space and time. Mass and motion at first glance are unrelated. However, if we want to describe the origin or root cause of such a motion, that is when we get into trouble. To answer this question adequately, we have to first understand what inertia is.

The concept of force goes back to ancient Greece, when Archemedes already seemed to grasp the vector nature of force with Eucledian geometry and trigonometry. Such methods were very effective at describing statics. The problem was that their understanding of kinetics was deeply flawed. Aristotle argued that for something to move, force has to be applied and the speed of an object's motion is proportional to the applied force and inversely proportional to the viscosity of the surrounding medium. It is obvious that they knew motion and force were related, but they simply had no clue what that somehow was.

The problem with this reasoning is that, well there are just so many that I don't even know where to start. It is Galileo's brilliant insight on inertia (Fig. 1.1) that allowed us to view the relationship between motion and force on a completely different light. He argued that every object has a natural tendency to maintain its motion, and a motion of the object will be unaltered unless there is net force acting on it. This tendency to maintain its motion is called inertia, and this is why we care about mass. But we will get to this point in the next chapter.

Another important contribution of Galileo is that he singled out constant velocity motion from all other motions, yet recognized that all constant velocity motion can be grouped as one indistinguishable set regardless of what that velocity is. From this, the concept of inertial frame was born.

Imagine a reference frame $\mathfrak{S}$ that we can declare as absolutely not moving. Within this frame, an object, say a hockey puck, is moving at a constant velocity v_p. An observer within the reference frame would see the motion of the hockey puck as free of any outside influence. Now imagine a moving frame $\mathfrak{S}'$ at a velocity $v_{S'}$ relative to the fixed frame $\mathfrak{S}$. Then to an observer within the $\mathfrak{S}'$ frame, the hockey puck slides at a velocity $v_p - v_{S'}$, which is also a constant

velocity. Thus the observer in the $\mathfrak{S}'$ frame would see the motion of the puck also free of outside influence. Despite the difference in velocity, both observers would see the effect of inertia, that is the hockey puck continues its original motion. For this reason, frames moving with a constant velocity is referred to as inertial frame. We will discuss the importance of this in more detail in the next chapter.

Chapter 2

Newton's Laws of Motion

With his concept of inertia, Galileo emphasized the importance of constant velocity motion. What Newton, in turn, did was to build upon that and emphasize the role of force, the physical quantity that is required to resist inertia. He then came up with three Laws of Motion. In its original form, they read:

Lex I: *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*

Lex II: *Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*

Lex III: *Actioni contrariam semper et qualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse quales et in partes contrarias dirigi.*

Now, in a language that we can understand, what they are saying are these:

Law I: *Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.*

Law II: *The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that impulse is impressed.*

Law III: *To every action there is always an equal and opposite reaction: or the forces of two bodies on each other are always equal and are directed in opposite directions.*

OK, this may not be so understandable either. So, we will delve into them one by one more carefully.

2.1 The First Law: The Law of Inertia

Law I: *Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impressed.*

All three laws that are seemingly stating different facts can be compressed into a single equation:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a} \tag{2.1}$$

This equation is called the equation of motion and it is, without a doubt, the single most important equation in classical mechanics. In fact, it can be said that this equation alone is all of classical mechanics. Throughout the year, we will learn a number of different applications of this single equation.

Then the question arises: why did of all the people, Newton, who probably knew better than anyone else that the laws can be compressed as a single equation, bothered to elaborate in such detail? For example, the first law seems to be, at first glance, a reiteration of the second law: from $\mathbf{F} = m\mathbf{a}$, it is quite obvious that when no force is applied ($\mathbf{F} = 0$), the object cannot accelerate nor decelerate.

However, it deserves to stand on its own for one very important reason we have already discussed in Chapter 1. It allows us to identify inertia and, equally importantly, set up a useful reference frame, the inertial frame. We can define inertial frame as

Definition: ***Inertial Frame*** is *a reference frame in which the First Law holds.*

As we shall soon see, only within the inertial frame, is the equation of motion meaningful.

Also, even though the first law seems to be a subset of more general second law, it is a very distinct subset in the sense that at zero acceleration, the concept of mass is completely meaningless. Because force-free motion is by definition, well, force-free, the object has no net-interaction with anything. Because the object is interactionless, you cannot distinguish a more massive object from a less massive object in this force free environment.

When you are staring at two different objects moving at two different velocities, it maybe tempting to conclude that one is lighter than the other because this is moving slower than that. However tempting it may be, you simply cannot.

Imagine you, your friend and two balls constitute the entire universe. You are sitting still (or so you think) and your friend is moving at a velocity of 5 m/s away from you to your right. Two balls are moving in opposite directions, a red one to your left at 1 m/s, a blue one to your right at 4 m/s. To you, a blue ball moves faster than the red. But for your friend, the red ball moves at 6 m/s and the blue one at 1 m/s, both to his left. For him, the red ball is moving faster.

Unless you are willing to accept the notion that two balls can have different mass for two different people, we now have a problem when you try to interpret physical world based on velocity. Velocity just allows you to establish relative reference frame against one another but the role of velocity stops right there (until you learn Einstein's relativity, but that is a whole new story).

2.2 The Second Law: The Equation of Motion

Law II: *The change of momentum of a body is proportional to the impulse impressed on the body, and happens along the straight line on which that*

impulse is impressed.

The first law hints at the concept of force, but the second law is the one that explicitly defines force. Mathematically expressing the original statement of the above Law II, we get

$$\mathbf{P} = \Delta\mathbf{p} \tag{2.2}$$

Newton appropriately defined momentum $\mathbf{p}$ of an object to be a quantity proportional to its velocity, *i.e.* $\mathbf{p} = m\mathbf{v}$ and an impulse $\mathbf{P}$ occurs when a force $\mathbf{F}$ acts over an interval of time Δt, *i.e.* $\mathbf{P} = \int_{\Delta t} \mathbf{F} dt$. Then the Eq. (2.2) can be rewritten as

$$\begin{aligned} \int_{\Delta t} \mathbf{F} dt &= \Delta(m\mathbf{v}) \\ \mathbf{F} = \frac{d(m\mathbf{v})}{dt} &= m\frac{d\mathbf{v}}{dt} = m\mathbf{a} \end{aligned} \tag{2.3}$$

Eq. (3.12) is where the equation of motion, Eq. (2.1) directly comes out of, and we can see that the net force applied to a body produces a proportional acceleration.

Also, the proportionality constant, m, between the velocity and momentum is of great significance here. From Eq. (2.1), we can see that for a given amount of force, acceleration is inversely proportional to m. In other words, the larger m is, the greater the tendency to stay its course of motion. To put it another way, m is the physical quantity that represents inertia *i.e.* mass.

From this definition of mass, we can see that measurement of force can be used to measure mass. A balance is an excellent example. Any balance uses balancing between the known amount of force and a force acting on an object of unknown mass, . *e.g.* gravitational force.

For what we will learn throughout this course, however, there is a more complicated use for the Newton's Second Law. If we assume that the mass m does not vary with time, Newton's equation of motion, $\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{r}}$, is simply a second-order differential equation that may be integrated to find $\mathbf{r} = \mathbf{r}(t)$ if the function $\mathbf{F}$ is known. Specifying the initial values of $\mathbf{r}$ and $\dot{\mathbf{r}} = \mathbf{v}$ then allows us to evaluate the two arbitrary constants of integration. We then determine the motion of a particle by the force function $\mathbf{F}$ and the initial values of position $\mathbf{r}$ and velocity $\mathbf{v}$.

2.3 The Third Law: The Law of Action and Reaction

Law III: *To every action there is always an equal and opposite reaction: or the forces of two bodies on each other are always equal and are directed in opposite directions.*

The third and final of Newton's laws is also known as the *action-reaction law*. In some sense, the first two laws were discovered already by Galileo and in fact none other than Newton himself gave credit to Galileo for the first law. However, the third law is the product of Newton's great insight that all forces are interactions between different bodies, and thus that there is no such thing

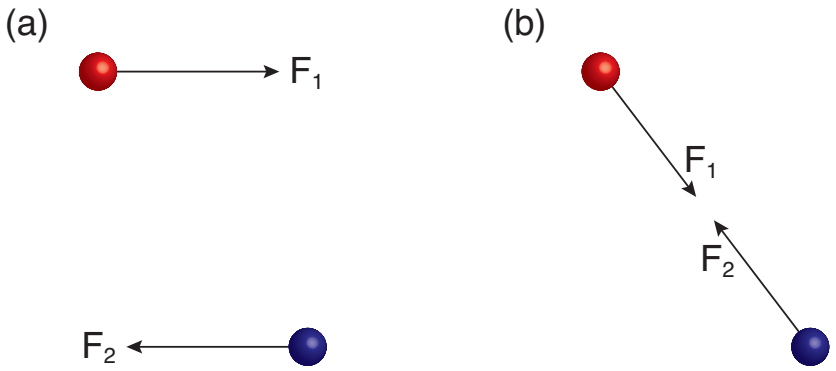

Figure 2.1: The Third Law of Motion in its (a) weak form and (b) strong form. In (a), there is no net force, but net torque is present.

as a unidirectional force or a force that acts on only one body. Whenever a first body exerts a force $\mathbf{F}_1$ on a second body, the second body exerts a force $\mathbf{F}_2 = -\mathbf{F}_1$ on the first body.

We can categorize the third law into two, a weak law of action-reaction and a strong law. In a weak form of the law, the action and reaction forces only has to be equal in magnitude and opposite in direction. They need not lie on a straight line connecting the two particles or objects. As a result, there could be a net torque acting on the system. On the other hand, the strong form of the law states that the forces have to be lined up. This distinction may be trivial, but it will become important in the next chapter.

For the time being, we will rewrite the third law, using the definition of force given by the second law,

$$\begin{aligned} \frac{d\mathbf{p}_1}{dt} &= -\frac{d\mathbf{p}_2}{dt} \\ m_1\mathbf{a}_1 &= -m_2\mathbf{a}_2 \\ \frac{m_2}{m_1} &= -\frac{a_1}{a_2} \end{aligned} \tag{2.4}$$

and from this we can give a practical definition of mass. If we set m_1 as the unit mass, then, by comparing the ratio of accelerations when m_1 is allowed to interact with any other body, we can determine the mass of the other body. So it is the Newton's Laws that allows us to define relative mass to our selection of unit mass m_1.

Of course, to measure the accelerations, we must have appropriate clocks and measuring rods. Let's now go back to IPK, IPM and IPS, these are basically the unit mass, a measuring rod and an appropriate clock required. From this, we can see that the job of BIPM and classical mechanics are not separable. Having IPM and IPS is what lets us to describe motion in a quantitative manner, and that in turn, according to the Newton's Laws, gives us the definition of mass.

Chapter 3

Laws of Conservation

In physics, there are a number of conservation laws, laws that state certain properties of a system does not change in time. Laws of conservation is related to differentiable symmetries of physical systems, as Emmy Noether pointed out, and can be a subject of intense study. However, we will not delve into this point too deeply, at least not yet, and will cover three widely used conservation laws that are direct consequences of Newton's Laws of Motion. These three are conservation of momentum, angular momentum and energy. We will look into these one by one.

3.1 Conservation of Momentum

Since the momentum is defined as the product of mass and velocity, for a system that conserves its mass, constant velocity is equivalent to constant momentum. Therefore, the concept of inertia stated in the First Law of Motion, in and of itself, is declaration of momentum conservation. Momentum conservation, however, extends beyond the force-free motion.

Imagine a particle that is accelerating. For this particle to accelerate, force has to be exerted. From the third law, when a force is acting on an object, there has to be an entity that is exerting the force, and the first object in question has to apply the same amount of force in opposite direction on to that entity. In a mathematical form: $\mathbf{F}_2 = -\mathbf{F}_1$.

The equation can be rewritten as follows:

$$\begin{gathered} \mathbf{F}_1 + \mathbf{F}_2 = 0 \\ \frac{d\mathbf{p}_1}{dt} + \frac{d\mathbf{p}_2}{dt} = \frac{d}{dt}(\mathbf{p}_1 + \mathbf{p}_2) = 0 \\ \therefore \mathbf{p}_1 + \mathbf{p}_2 = \text{constant} \end{gathered} \tag{3.1}$$

In other words, in the absence of external force, the total momentum of the two objects that are exchanging force is constant. The same logic can be applied to a multi-object system beyond two objects. It follows that, at least classical mechanically, total momentum of the universe is conserved. However, this is not a practically useful statement. What this implies is that an external force on a system is always an internal force of a larger system, and in reality, studying mechanics comes down to isolating an observable systems appropriately.

3.2 Conservation of Angular Momentum

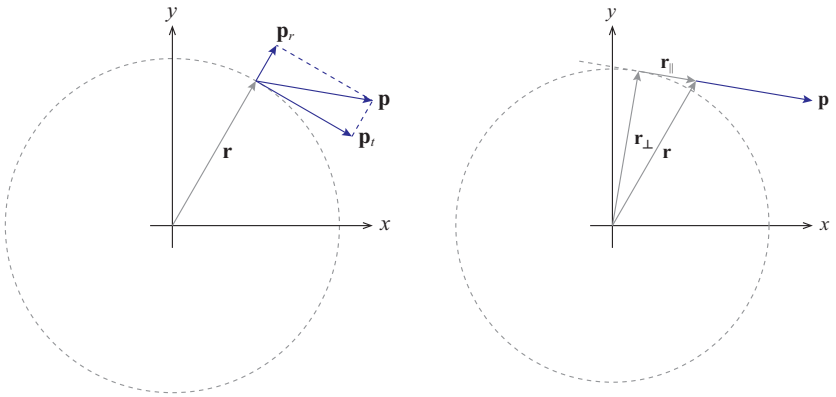

Figure 3.1: Visually interpreting angular momentum.

For any moving particle, one can define angular momentum, a representation of rotational motion, with respect to a reference point in space as

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{3.2}$$

where $\mathbf{r}$ is the position of the particle with respect to the reference point, and $\mathbf{p}$ is the linear momentum of the particle.

One thing that should be emphasized at this point is that, unlike linear momentum, angular momentum has to be defined about a reference point. Once the reference point is defined, any linear motion can be decomposed into two components, a radial component, $\mathbf{p}_r$, and a tangential component, $\mathbf{p}_t$, around a circle with radius r about the reference point as shown in Fig. 3.2(a). Then the definition of angular momentum is equivalent to

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}_t \tag{3.3}$$

An alternative is to decompose the position vector into a component that is parallel to the momentum, $\mathbf{r}_\parallel$ and a perpendicular component, $\mathbf{r}_\perp$ as shown in Fig. 3.2(b). Breaking down the position vector yields to

$$\mathbf{L} = \mathbf{r}_\perp \times \mathbf{p} \tag{3.4}$$

From these relations, we can tell that angular momentum is a quantity that represents "with how much momentum around how big a circle is the particle moving?"

This creates a rather unique situation to which there is no analogy in a linear motion. For a given linear motion, time dependence of momentum is identical regardless of the (inertial) reference frame. However, with an angular motion the reference frame becomes extremely important. Imagine a particle going around a circle at a constant speed. If the reference frame is fixed at the center of the circle as shown in Fig. 3.2(a), the angular momentum is constant. If the reference frame is off the center of the circle, the angular momentum of the system is no longer constant, as is evident in Fig. 3.2(b). To resolve this discrepancy, we need to introduce a new physical quantity, namely torque.

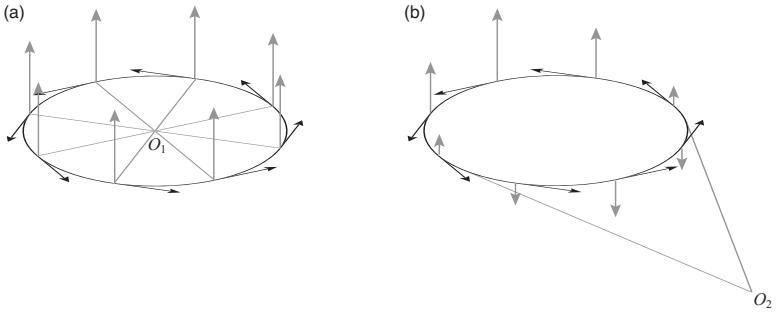

Figure 3.2: Angular momentum of an identical motion in two different inertial reference frames. The black arrows indicate the linear velocity of the object in motion. The grey arrows indicate the angular momentum with respect to the frame of reference noted as O_1 and O_2.

We will define torque acting on a single particle as the moment of force:

$$\mathbf{K} \equiv \mathbf{r} \times \mathbf{F} \tag{3.5}$$

which represents the amount of force applied to the particle with respect to some reference point.

When there is more than one particle in the system, the total angular momentum of the system is a sum of angular momenta of individual particles, and the total torque on the system is a sum of torques on each particle.

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i \tag{3.6}$$

$$\mathbf{K} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{tot}} \tag{3.7}$$

If we take time derivative of the angular momentum,

$$\frac{d\mathbf{L}}{dt} = \sum_i \left(\frac{d\mathbf{r}_i}{dt} \times \mathbf{p}_i + \mathbf{r}_i \times \frac{d\mathbf{p}_i}{dt} \right) \tag{3.8}$$

Since $\mathbf{p}_i = m\frac{d\mathbf{r}_i}{dt}$, the first term in Eq. (3.8) is zero. From Newton's Second Law, $\frac{d\mathbf{p}_i}{dt}$ is the net force acting on the i-th particle, *i.e.* $\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i^{\text{tot}}$. Thus we get the relation

$$\mathbf{K} = \frac{d\mathbf{L}}{dt} \tag{3.9}$$

Let us break this relation down a bit more. In a multi-particle system, the net force imposed on a particle is the sum of external force and all the internal forces among the constituent particles,

$$\mathbf{F}_i^{\text{tot}} = \mathbf{F}_i + \sum_j \mathbf{G}_{ij} \tag{3.10}$$

where $\mathbf{F}_i$ and $\mathbf{G}_{ij}$ are the external force applied on the particle and the internal force on the i-th particle from the j-th particle, respectively. Combining this

result with the Newton's Third Law, *i.e.* $\mathbf{G}_{ij} = -\mathbf{G}_{ji}$, Eq. (3.8) can be rewritten as

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_{i,j} \mathbf{r}_i \times \mathbf{G}_{ij} \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_{i<j} (\mathbf{r}_i \times \mathbf{G}_{ij} + \mathbf{r}_j \times \mathbf{G}_{ji}) \\ &= \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_{i<j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{G}_{ij} \\ &= \sum_i \mathbf{K}_i + \sum_{i<j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{G}_{ij} \end{aligned} \tag{3.11}$$

The second term in the Eq. (3.11) is an internal torque. What is interesting here is that the internal torque vanishes only if either there are no internal forces $\mathbf{G}_{ij} = 0$ or $\mathbf{r}_i - \mathbf{r}_j$ is in the same direction as $\mathbf{G}_{ij}$, that is the internal forces are along the lines connecting the two particles. In other words, if the internal forces are present, the internal torque vanishes only if the strong form of Newton's Third Law holds. For almost all the problems that we will deal with throughout this course, the strong form actually holds, so you need not to worry about this subtle distinction.

In such cases, the total angular momentum of a particle system does not change with time in the absence of an external torque. In other words, the angular momentum is conserved.

3.3 Conservation of Energy

Conservation energy is one of the fundamental conservation laws of nature. It should be a concept you must have learned since middle school and be familiar with by now. However, the fact that energy, unlike momentum or angular momentum, takes many forms makes the law of conservation of energy not as trivial as one might think at first. In light of these facts, it should come as no surprise that Newton never mentioned energy in his work.

Nonetheless, one cannot simply look over the importance of the energy conservation. Within the framework of classical mechanics, conservation of energy refers to lossless transfer of energy between two very specific forms of energy: kinetic energy and potential energy. So the first step here would be to define these two types of energy.

3.3.1 Kinetic energy

Let us start from the Second Law of Motion:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} \tag{3.12}$$

We need not define or restrict the type of force $\mathbf{F}$, and it can be considered simply as the net force acting on the particle of interest. Take the scalar product of Eq. (3.12) with $\mathbf{v}$, and we get

$$\mathbf{F} \cdot \mathbf{v} = m\frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = \frac{d}{dt}\left(\frac{1}{2} m\mathbf{v} \cdot \mathbf{v}\right) \tag{3.13}$$

By defining $T = \frac{1}{2}mv^2$, we can rewrite the above equation as

$$T_2 - T_1 = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} dt \tag{3.14}$$

When $\mathbf{F}$ is a force field $\mathbf{F}(\mathbf{r})$, that is the amount of force $\mathbf{F}$ acting on the particle is given for each position $\mathbf{r}$, the right hand side of the Eq. (3.14) becomes

$$\int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} dt = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt} dt = \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = W_{12} \tag{3.15}$$

which is the work done by moving the particle across the fixed path $\mathcal{C}$ from point $\mathbf{r}_1$ to $\mathbf{r}_2$ during the time between t_2 and t_1. (See Fig. 3.3.1.) Since the right hand side of the Eq. (3.14) represents work done on a particle, the left hand side must represent a change in energy in the form of $T = \frac{1}{2}mv^2$. Because this energy T represents the energy stored in a particle's motion, that is T depends on m and $\mathbf{v}$, we define T as the kinetic energy, the energy of motion.

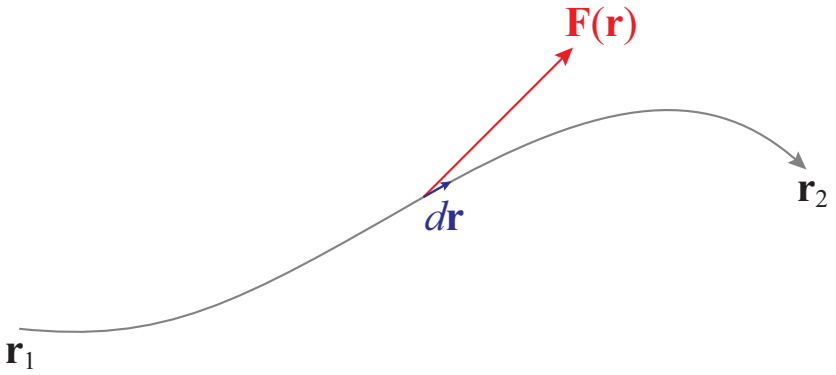

Figure 3.3: Work done on a particle can be calculated through a line integral of $\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ over a path $\mathcal{C}$ of the motion from point $\mathbf{r}_1$ to point $\mathbf{r}_2$.

3.3.2 Potential energy

If the force field $\mathbf{F}(\mathbf{r})$ can be expressed in the form

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \tag{3.16}$$

where $V(\mathbf{r})$ is a scalar function only of position, then $\mathbf{F}$ is said to be a conservative field. In such a case, Eq. (3.14) is reduced to

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= -\int_{\mathcal{C}} \nabla V \cdot d\mathbf{r} \\ &= -\int_{\mathcal{C}} \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \\ &= -\int_{\mathcal{C}} dV = V(\mathbf{r}_1) - V(\mathbf{r}_2) \end{aligned} \tag{3.17}$$

This is a striking result. This states that when the force field is conservative, the work done on a particle by the force field across the path $\mathcal{C}$ is simply the

difference between a scalar field $V(\mathbf{r})$ at the initial and final points of the path $\mathcal{C}$. The function $V(\mathbf{r})$ is a form of energy that is stored in its position, and is usually referred to as the potential energy.

3.3.3 Mechanical energy conservation

From the relation between the kinetic energy and work, and the potential energy and work, we can find the following relationship for a particle under the influence of a conservative force:

$$W_{12} = T_2 - T_1 = V_1 - V_2 \tag{3.18}$$

and rewriting this equation, we get

$$T_1 + V_1 = T_2 + V_2 \tag{3.19}$$

Because the choice of initial and final time and position, t_1, t_2, $\mathbf{r}_1$ and $\mathbf{r}_2$ is arbitrary, the above equation is equivalent to saying that the mechanical energy, *i.e.* the sum of kinetic and potential energy, is constant in time:

$$T + V = E = \text{constant} \tag{3.20}$$

Simply put, the mechanical energy is conserved for a particle moving in a conservative force field.

An important and very useful property of this energy conservation is that because the potential energy is a function only of the position of the particle, the path it follows becomes irrelevant. Put in another way, even if the particle is under a geometrical constraint, *e.g.* a pendulum attached to a string or a marble rolling down a guide rail, if the constraint forces on the particle do no work on the particle, the mechanical energy is still conserved.

Chapter 4

Solving Equation of Motions

The starting point of classical mechanics is the equation of motion given by

$$\mathbf{F} = m\mathbf{a} \tag{4.1}$$

Since, at the end of the day, what we want to find out in classical mechanics is time evolution of position of a physical object, $\mathbf{r}(t)$, the above equation turns out to be a differential equation of the form

$$m\frac{d^2\mathbf{r}}{dt^2} - \mathbf{F}(\mathbf{r}, t) = 0 \tag{4.2}$$

In some rare occasions, the force $\mathbf{F}(\mathbf{r}, t)$ may contain higher order derivatives of $\mathbf{r}$ with respect to t. But, as I said, this is very rare, especially in undergraduate physics level, and for the most part, the equation of motion is a second order differential equation.

If $\mathbf{F}(\mathbf{r}, t)$ takes the form

$$\mathbf{F}(\mathbf{r}, t) = f_2(t)\frac{d^2\mathbf{r}}{dt^2} + f_1(t)\frac{d\mathbf{r}}{dt} + f_0(t)\mathbf{r} + f(t) \tag{4.3}$$

we can rewrite Eq. (4.2) as

$$a_2(t)\frac{d^2\mathbf{r}}{dt^2} + a_1(t)\frac{d\mathbf{r}}{dt} + a_0(t)\mathbf{r} = f(t) \tag{4.4}$$

where $a_2(t) = m - f_2(t)$, $a_1(t) = -f_1(t)$ and $a_0(t) = -f_0(t)$. Aside from the fact that it is a vector equation, Eq. (4.4) has the same form as a linear n-th order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \tag{4.5}$$

where $a_i(x)$'s and $f(x)$ depend only on x.

4.1 Force-Free Motion

Let us first consider a one dimensional force-free motion with $F(x,t) = 0$. From Eq. (4.2), we get the equation of motion

$$m\frac{d^2x}{dt^2} = 0 \tag{4.6}$$

By defining $v \equiv dx/dt$, the equation reduces to a first order differential equation

$$\frac{dv}{dt} = 0 \tag{4.7}$$

A fairly straightforward integration yields the following result

$$v = \int \left(\frac{dv}{dt}\right) dt = \int dv = v_0 \tag{4.8}$$

Since $v = dx/dt$, we can rewrite the above equation as

$$\frac{dx}{dt} = v_0 \tag{4.9}$$

Following the same procedure as Eq. (4.8), we get

$$x = \int \left(\frac{dx}{dt}\right) dt = \int v_0 dt = v_0 t + x_0 \tag{4.10}$$

Here, x_0 and v_0 are unspecified constants. The position changes with time along a straight line as shown in Fig. 4.1.

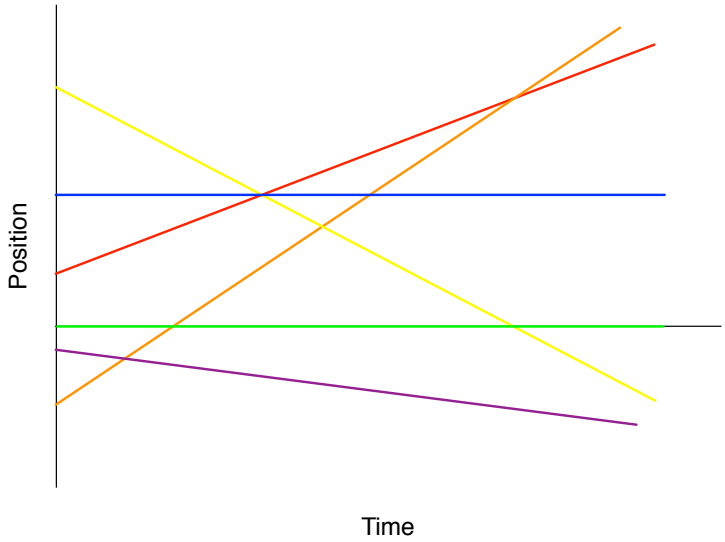

Figure 4.1: One dimensional force-free motion.

In this case, its starting point (y-intercept on the graph) and velocity (slope) are unknown, however, if the initial conditions x_0 and v_0 are specified, we can uniquely pick out a single line among infinitely many possibilities.

This method can be generalized to higher dimensional space, and one will get a solution of the form

$$\mathbf{r} = \mathbf{v_0}t + \mathbf{r}_0 \tag{4.11}$$

In this case, although unfortunate choice of coordinate system might force you to have to solve a set of three identical differential equations for this problem, in fact, an appropriate choice of coordinate system can always reduce the problem to a one dimensional problem.

4.2 Constant Force Motion

We now turn on a constant external force, $\mathbf{F}(\mathbf{r}, t) = \mathbf{F}_0$ on to this particle. But, we will once again start with one dimensional motion.

4.2.1 Constant force motion in one dimension

One dimensional constant force motion can be expressed by the equation of motion:

$$\frac{d^2x}{dt^2} = \frac{F_0}{m} = a_0 \tag{4.12}$$

Same method used in the previous section can be utilized to find the solution to this equation, and one can easily find that the solution is

$$x = \frac{1}{2}a_0t^2 + v_0t + x_0 \tag{4.13}$$

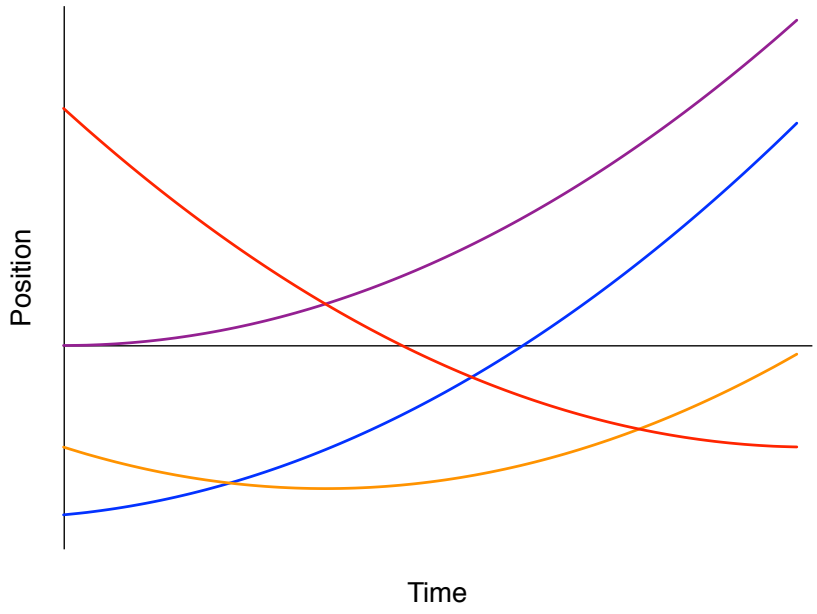

Figure 4.2: A general solution to one dimensional constant force motion.

The solution has an additional term $\frac{1}{2}a_0t^2$, which is the effect of acceleration due to the force. The solution is no longer a straight line. Instead, we get a

parabola and if the initial condition is not specified, the parabola can take any of the curves in Fig. 4.2.1. Here, note that the acceleration a_0 is fixed by the force applied to the given mass in the problem, $a_0 = F_0/m$ and is not affected by the initial values of v_0 and x_0.

We can rewrite the Eq. (4.13)

$$x = \frac{1}{2}a_0(t + \frac{v_0}{a_0})^2 + x_0 - \left(\frac{v_0}{a_0}\right)^2 \tag{4.14}$$

which is an equation obtained by moving $x = \frac{1}{2}a_0t^2$ by $-\frac{v_0}{a_0}$ along the time-axis and by $x_0 - \left(\frac{v_0}{a_0}\right)^2$ along the position-axis. From this perspective, the initial value problem simply becomes at what point along the parabolic space-time trajectory you start observing the system as shown in Fig. 4.2.1.

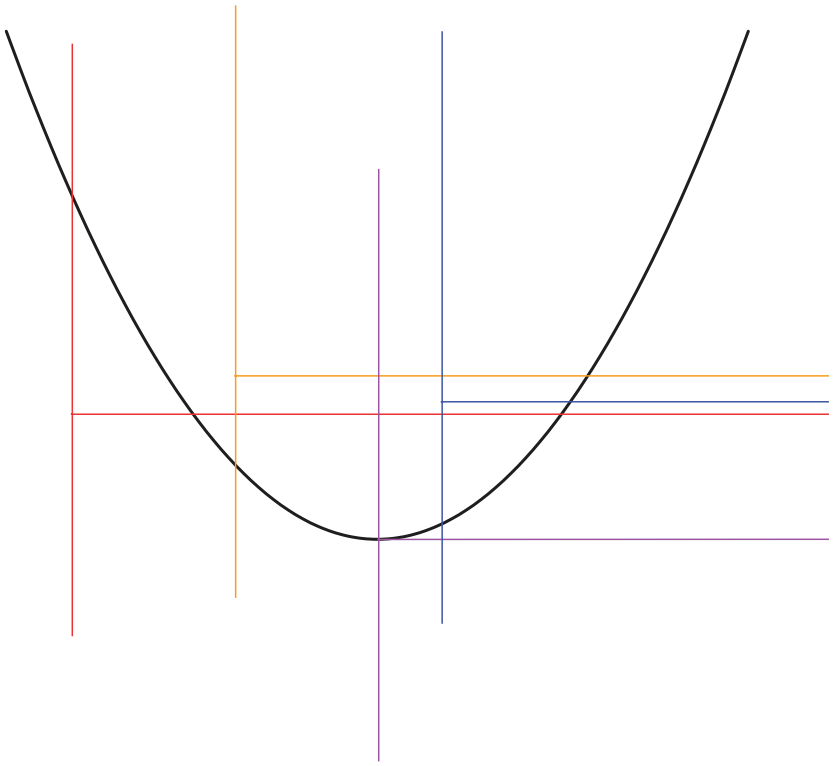

Figure 4.3: A different way of interpreting the initial value problem of one dimensional constant force motion.

4.2.2 Constant force motion in two dimensions

The two dimensional problem with constant force is a bit trickier than the force-free case where one could always, in principle, reduce the problem to a one dimensional problem. Here, one has to examine the direction of the force (acceleration) and the direction of the initial velocity carefully.

If the velocity vector and acceleration vectors line up, either in the same direction or opposite direction, as in Fig. 4.3.1(a), the problem can be reduced

to a one dimensional problem. This is because acceleration is defined as the change of velocity with respect to time and when the velocity component is all in the direction of acceleration, the particle velocity always remains in that direction, and we need not care about the motion in any other directions. Note that the problems with zero initial velocity also falls into this category. Such problems include the case of a bullet shot straight up in the air, or a free fall problem.

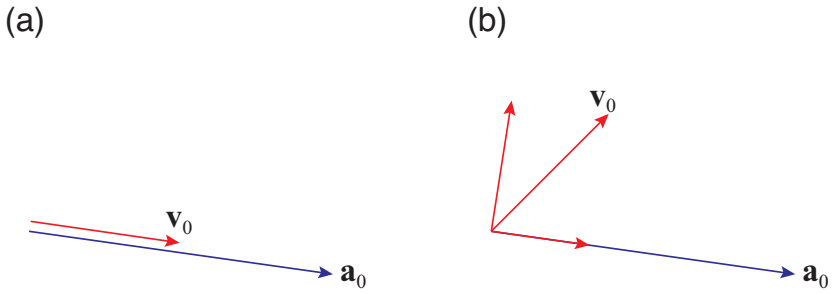

Figure 4.4: Solving a two dimensional constant force problem.

On the other hand, if the velocity and acceleration do not line up, then we can divide the velocity into two components, one along the direction of acceleration and the other perpendicular to the acceleration. Then we end up with two different equations. Along the direction of acceleration, we have to solve for a problem of constant force motion, whereas in the perpendicular direction, the motion is force-free.

For example, assume that the direction of acceleration is along the y-axis, and there is no acceleration along the x-axis. Then the motion along the x-axis can be described with the Eq. (4.10).

$$x = v_{x0}t + x_0 \tag{4.15}$$

and the motion along the y-axis with the Eq. (4.13)

$$y = \frac{1}{2}a_0t^2 + v_{y0}t + y_0 \tag{4.16}$$

where v_{x0} and v_{y0} are the initial velocity along the x- and y-axis, respectively. The vector $\mathbf{r} = (x_0, y_0)$ marks the initial position. Combining Eq. (4.15) and Eq. (4.16), we get

$$\begin{aligned} y &= \frac{1}{2}a_0\left(\frac{x - x_0}{v_{x0}}\right)^2 + v_{y0}\left(\frac{x - x_0}{v_{x0}}\right) + y_0 \\ &= \frac{1}{2}\frac{a_0}{v_{x0}^2}x^2 + \left(\frac{v_{x0}v_{y0} - a_0x_0}{v_{x0}^2}\right)x + \left(\frac{1}{2}\frac{a_0}{v_{x0}^2}x_0^2 - \frac{v_{y0}}{v_{x0}}x_0 + y_0\right) \end{aligned} \tag{4.17}$$

This is what we call a trajectory. The equation spatially traces how the particle moves without specifically knowing where the particle would be at a given time. This is a useful way of visualizing a motion only in two or higher dimensions. (In one dimensional motion, the trajectory is always a straight line and is thus boring.) From this, one can see that the trajectory is a parabolic equation for

a general constant force motion, and this is exactly what we see for a baseball thrown in the air or a fired canonball with an arbitrary angle.

It is interesting to note that, just as any force-free motion could be reduced to a one dimensional problem with an appropriate choice of coordinates, any constant force motion can be reduced to a two dimensional motion. This is because two vectors, $\mathbf{a}_0$ and $\mathbf{v}_0$ form a two dimensional plane and thus no motion can exist outside the plane.

4.3 Varying Force Motion

In this section, we will finally deal with forces that are not constant in time. However, we will not deal with forces that changes explicitly with time. Instead, we will look at the problems with forces that depend on velocity or position. Because we are studying mechanics, that is a field of physics where we inherently deal with particles in motion, forces that depend on velocity or position results in implicit change in their magnitudes or directions with time.

4.3.1 Drag force

Anyone who has swum or rowed a boat understands that motion in water is considerably more resistive than the motion in air. This is due to the drag force of a medium and the concept of drag force goes back no later than ancient Greece, when Aristotle argued that the speed of an object's motion is inversely proportional to the viscosity of the surrounding medium. As we have mentioned earlier, Aristotle's world view on mechanics was deeply flawed, however, his observation that the viscosity is related to particle's speed contains elements of truth.

The difficulty in estimating the drag force lies in understanding the microscopic origin of it. When an object passes through a medium, the object has to push the medium out of its way. It requires force acting on the medium, which in turn creates force acting back on the object. To theoretically derive how this reaction of the medium turns into the net force acting on the moving object, it requires full understanding of interaction among particles constituting the medium and between the particles and the moving object. This is an extremely difficult task, and our understanding of the drag force, for the most part, relies on phenomenological analysis. Even this phenomenological analysis is well outside the scope of this course, and we will just take the result of such analysis here.

An ideal particle with zero cross section does not have its place in a phenomenological description of mechanics, so we will assume an object with maximum cross sectional radius a perpendicular to the direction of its motion. If the velocity of the object is given by $\mathbf{v} = v\mathbf{e}_v$, the drag force acting on the object is given by

$$\mathbf{D} = -F(R)\rho a^2 v^2 \mathbf{e}_v \tag{4.18}$$

where $F(R)$ is a single variable function of R whose value must be obtained phenomenologically. ρ is the density of the medium and $\mathbf{e}_v$ is the unit vector along the direction of motion. The minus sign signifies that the drag force is in the opposite direction to the motion, causing the resistance. R is what is known

as the Reynolds number named after Osborne Reynolds, a 19th century expert in fluid dynamics, and is defined as

$$R \equiv \frac{\rho a v}{\mu} \tag{4.19}$$

where μ is the viscosity of the medium. Its exact definition and meaning is, once again, outside the scope of this course and a subject of more detailed fluid dynamics. We will just state here that the function $F(R)$ does not strongly depend on R over a wide range of values (about $1000 < R < 100,000$), and thus can be treated as a constant close to unity. Therefore, for an object moving through a medium that satisfies the condition of $1000 < R < 100,000$, the drag force can be represented as

$$\mathbf{D} \approx -\rho a^2 v^2 \mathbf{e}_v \tag{4.20}$$

(Exact value of $F(R)$ depends on the geometry of the object and other factors, but for the purpose of this class, we will treat it as unity.) If a particle with mass m happens to have a velocity $\mathbf{v}$ in the medium, and the only force acting on it is the drag force caused by its motion, the equation of motion can be written as

$$m\frac{d^2\mathbf{r}}{dt^2} = m\frac{d\mathbf{v}}{dt} = \mathbf{F} = \mathbf{D} = -\rho a^2 v^2 \mathbf{e}_v \tag{4.21}$$

Because the drag force is proportional to the square of velocity, this is called quadratic law of resistance. Here the direction of the force and the direction of the velocity is matched and thus we can reduce the problem into one dimension

$$m\frac{dv}{dt} = -\rho a^2 v^2 \tag{4.22}$$

The equation contains the square of velocity v^2, and is a non-linear differential equation. Fortunately, this equation turns out to be one of a few cases where non-linear differential equation can be solved. We will leave it as an exercise. The final result turns out to be

$$v(t) = \frac{v_0}{1 + t/\tau} \tag{4.23}$$

$$x(t) = v_0 \tau \ln(1 + t/\tau) + x_0 \tag{4.24}$$

where $1/\tau = v_0 \rho a / m2$. What is remarkable here is that the velocity of this object reaches zero as time goes on, however, the particle nonetheless travels across infinite distance. This is because the drag force quite large at large velocities, but drops rapidly as the particle slows down due to the drag. The drag force becomes so small at small velocities that the velocity cannot quite come to zero fast enough, and the particle never comes to a complete stop. This, however, seems unphysical as no object can travel through a medium for indefinite distances.

A closer inspection of the condition required for this problem solves this dilemma. The equation holds for a finite range of the Reynolds number and from the way Reynolds number is defined, $R = \rho a v/\mu$, one can see that the quadratic law of resistance cannot hold as the velocity reaches zero. It is reasonable to think that the density and viscosity of the medium and the size of the object traveling through the medium does not change with the object's velocity, and thus R must approach zero as v approaches zero.

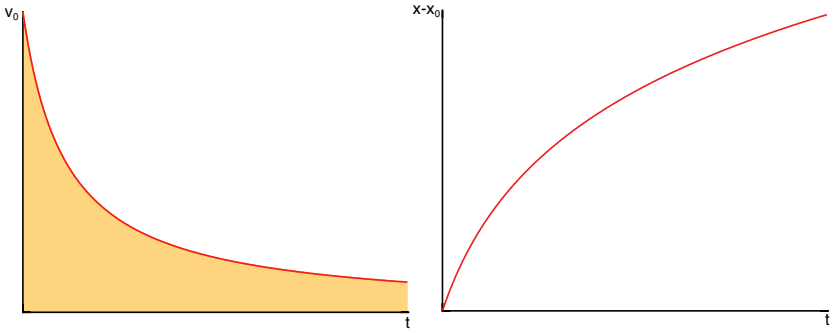

Figure 4.5: Velocity and position evolution of the motion under the quadratic law of resistance. The area under the velocity curve (light orange region) represents the distance traveled by the object.

It turns out, in small R limit, the $F(R)$ has $1/R$ dependence. A more careful analysis shows that $F(R)$ reaches a value close to $6\pi/R$, and the drag force is then represented by the equation

$$\mathbf{D} \approx -6\pi a\mu\mathbf{v} \tag{4.25}$$

Because the drag force changes linearly with velocity, this is called the linear law of resistance. Again, solving this equation will be left as an exercise. The final result in this case is

$$v(t) = v_0 e^{-t/\tau} \tag{4.26}$$
$$x(t) = v_0\tau(1 - e^{-t/\tau}) + x_0 \tag{4.27}$$

where $1/\tau = 6\pi a\mu/m$. In this case, the velocity decays exponentially, and as a result the traveling distance is also limited to a finite value of $\Delta x = v_0\tau$, asymptotically approaching it. In other words, a particle moving in a linearly resistive medium will slowly come to a stop after traveling the distance equal to the product of the initial velocity, v_0, and the time constant, τ inversely proportional to the viscosity.

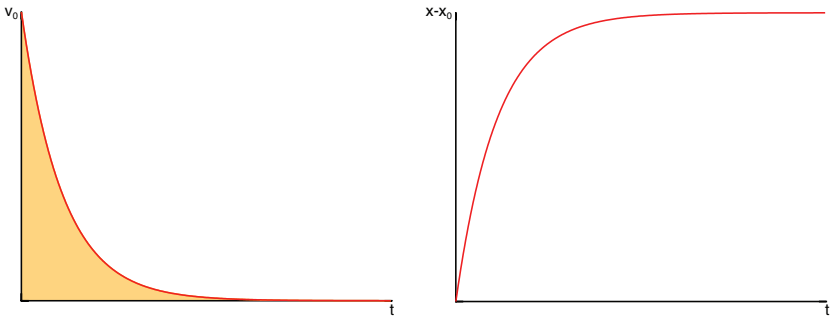

Figure 4.6: Velocity and position evolution of the motion under the linear law of resistance. The area under the velocity curve (light orange region) represents the distance traveled by the object.

We can now think of a case where an object is pushed through a viscous medium. We will here consider the linear resistance, and leave the quadratic

resistance case as an exercise. One such example is a falling object through a viscous medium such as air. The push, here, is the gravitation, and when the object is in motion, it will experience a drag force in addition to gravity. We will set the upward direction as positive direction, so the gravitational force is then $F_g = -mg$. The drag force is always in the opposite direction of the direction of the motion and thus $D = -mv/\tau$. So the equation of motion is

$$\begin{aligned} m\frac{dv}{dt} &= -mg - \frac{mv}{\tau} \\ \frac{dv}{dt} &= -g - \frac{v}{\tau} \end{aligned} \tag{4.28}$$

Solve the equation for initial values v_0 and x_0, and we get

$$\begin{aligned} v(t) &= (v_0 + g\tau)e^{-t/\tau} - g\tau \\ x(t) &= x_0 + (v_0\tau + g\tau^2)(1 - e^{-t/\tau}) - g\tau t \end{aligned} \tag{4.29}$$

Here, the final velocity approaches $v = -g\tau$, which is the velocity at which the net force becomes zero, that is

$$F_g + D = -mg - \frac{mv}{\tau} = 0 \tag{4.30}$$

This is what is known as the terminal velocity of a motion in a drag medium.

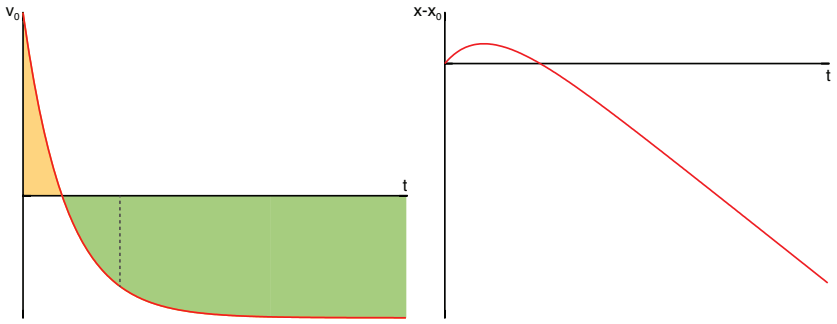

Figure 4.7: Velocity and position evolution of the motion under a constant force plus linear resistance. The area under the velocity curve (light green region subtracted from light orange region) represents the distance traveled by the object. The grey dashed line marks the point in time at which the particle returns to its original position. Note that the velocity curve approaches non-zero value due to the shift in the force equilibrium point. Accordingly, the position of the object approaches a straight line with non-zero slope representing a constant velocity motion. Because the terminal velocity is negative in this case, the slope is also negative.

Suppose there is no drag force. Then the motion will indefinitely accelerate, provided that the object does not hit the wall or ground. Since there is a force opposing the motion that grows with velocity, that acceleration has to stop once the velocity reaches the point where the constant pushing (or pulling) of the object matches the drag force. At this point, there is no net force and the inertia takes over and the object continues to move at the speed of $g\tau$. Therefore,

instead of asymptotically approaching zero velocity as in the no outside force case, the object approaches the terminal velocity. It is notable that in this case the velocity does not oscillate around the terminal velocity, but approaches asymptotically.

4.3.2 Harmonic oscillator

There are two types of position dependent forces that play an important role in introductory classical mechanics. One is the gravitational force that is inversely proportional to distance and the other is the restoring force that is responsible for harmonic oscillation. Such restoring force follows Hooke's law , *i.e.* force is proportional to the displacement. We will take a closer inspection of them in Chapter 8 and Chapter 11, respectively. For now, we will briefly look at the simple harmonic motion in one dimension.

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F = -kx \\ \frac{d^2x}{dt^2} &+ \omega^2 x = 0 \end{aligned} \tag{4.31}$$

where $\omega^2 = k/m$. This is a second order linear differential equation with constant coefficients, thus the solution should take the form $x(t) = e^{\lambda t}$. We get auxiliary equation

$$\lambda^2 + \omega^2 = 0 \tag{4.32}$$

and the general solution is then

$$x(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} \tag{4.33}$$

where A_1 and A_2 are arbitrary complex numbers that should be determined by the initial condition. By representing A_1 and A_2 with a set of amplitude and phase,

$$A_1 = Ae^{-i(\phi+\pi/2)} \qquad \text{and} \qquad A_2 = Ae^{i(\phi-\pi/2)} \tag{4.34}$$

we can rewrite the solution given by Eq. (4.33) as

$$x(t) = A\cos(\omega t - \phi) \tag{4.35}$$

This is an oscillatory solution, and one can easily check that the solution satisfies the Eq. (4.31). Here the amplitude, A, and the phase ϕ are two arbitrary constants that should be determined by the initial condition. The phase ϕ determines during what part of the oscillatory cycle, one's observation begins. For example, if the initial condition is given by $x_0 = 0$ and $v_0 = A_0\omega$, the amplitude and the phase are fixed to be $A = A_0$ and $\phi = \pi/2$. Then the solution is a simple sine function $x(t) = A_0 \sin\omega t$.

What is interesting about this motion is that there is a force equilibrium point at $x = 0$. Unlike the drag force case where the velocity approaches its force equilibrium point asymptotically, here the position oscillates around the force equilibrium point. This is an interesting point to ponder about, and I will leave it to the readers to think about it.

Chapter 5

Lagrangian Mechanics

In essence, we have covered everything we need to know in classical mechanics, *i.e.* equation of motion, in the previous chapters. Most of the readers, however, would feel that most if not all of the stuff we've learned so far is not set apart from things taught in the mechanics part of the freshman physics. We can look at some very sophisticated examples, but we will put it off to later chapters. Instead, we will develop an entirely new approach commonly known as analytical mechanics. We will start with configuration space and generalized coordinates.

5.1 Configuration Space

Imagine applying a constant force F_0 at time $t = 0$ on a particle with mass m at rest. It will accelerate at a constant rate $a = F_0/m$ and we can visualize the motion in our head. Now imagine another particle with mass m attached to a massless spring with spring constant α. The spring is compressed by some distance A, and then being released at $t = 0$. The second particle will follow harmonic oscillation with the oscillation frequency $\omega = \sqrt{\alpha/m}$. This is also not very difficult to visualize in our head. Graphical representations of these motions are given in Fig. 5.1.

Here, we considered two different particles moving in one dimension. However, we can also represent a motion of a single particle moving in two dimensions with the same set of graphs. In general, each dimensional motion for each particle can be represented in a *time-position space*.

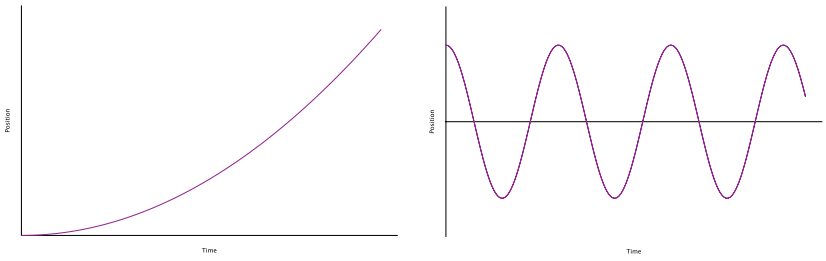

Figure 5.1: Graphical representations of constant force motion and harmonic oscillation.

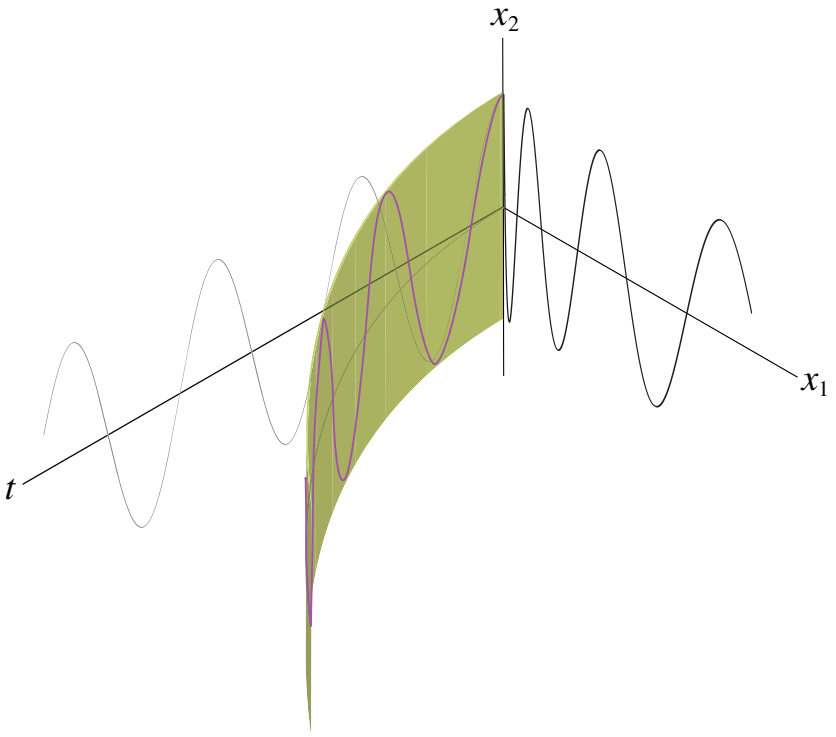

Figure 5.2: Graphical representations of constant force motion and harmonic oscillation.

An alternative representation of these motions is to present them simultaneously. Since the time axis is common in both graphs, we can merge the time axes of the two graphs. Then, instead of plotting the graphs in two separate *time-1D position space*, we can construct an abstract *time-2D position space* as shown in Fig. 5.1.

Projection of the purple trace in the figure onto t-x_1 space or t-x_2 space yields the two graphs in Fig. 5.1. The projection of the trace on to x_1-x_2 space yields a graph $x_2 = A\cos\beta\sqrt{x_1}$ where $\beta = \sqrt{2\alpha/F_0}$. A simultaneous motion of the two particles can be obtained by moving along the curve on x_1-x_2 space as time passes on.

We can expand this idea to many more particles and also into more than one dimensional motions. As we stated earlier, each dimensional motion of each particle can be plotted within a *time-position space*. If there are N particles in three dimensional space, we get $3N$ such graphs. We can merge time axes of all these $3N$ graphs into a single identical time axis. Then we can think of a single point moving in this $3N + 1$-dimensional space to represent the motions of all the particles in all directions. We can also making a projection on to the $3N$-dimensional space by compressing the time axis, and get a curve that represents the motion. This $3N$-dimensional space has a special name, and it is usually called *configuration space*.

The curves in configuration space, however, does not define a motion uniquely. Consider two different motions shown in Fig. 5.1. Two motions have different

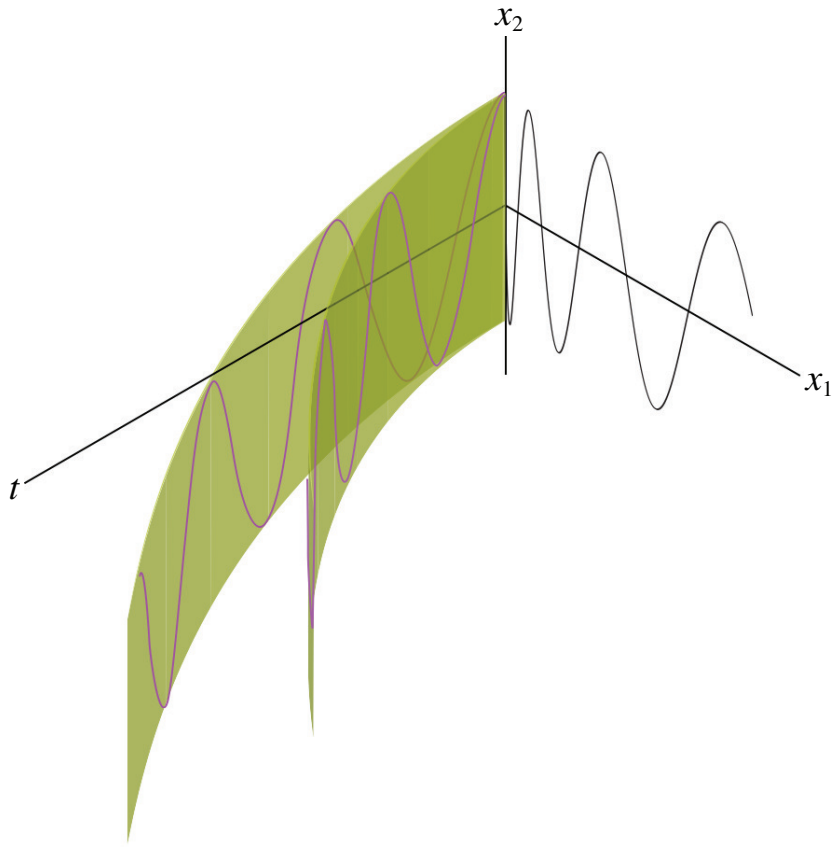

Figure 5.3: Graphical representations of constant force motion and harmonic oscillation.

values of F_0 and α but identical F_0/α. The general trend of the motions look similar, and in fact, the curves that represent two motions in the configuration space are identical. The only difference is that one motion is slower than the other by a constant factor. We will discuss the shortcomings of the configuration space in more detail later. In the mean time, we will develop an analytical mechanics commonly known as Lagrangian mechanics.

5.2 Lagrangian Equations of Motion

There are many minimization principles in physics, most widely known case being the Fermat's principle of least time. It states that the path taken between two points by a ray of light is the path that can be traversed in the least time. To calculate the actual path taken by light between two points based on this minimization principle the Euler-Lagrange equation based on the calculus of variation is quite useful. The derivation of the Euler-Lagrange equation is given in the appendix and here we will just take the end result: finding a function

$x(t)$ that minimizes the following integral

$$J[x] = \int_a^b F(x, \dot{x}, t)dt \tag{5.1}$$

is equivalent to solving the following differential equation

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \tag{5.2}$$

Now, how does this pertain to the mechanical problems that we are interested? In mechanics, there is a minimization principle usually referred to as Hamilton's principle, and from this principle, we can obtain Lagrangian equations of motion. Why a mechanical analysis followed from *Hamiltonian* principle is called *Lagrangian* mechanics, not Hamiltonian mechanics, may be puzzling. It is related to the historical development of Lagrangian mechanics, that is, Lagrangian mechanics in its original form did not necessarily evolve based on calculus of variation.

Lagrange was mainly interested in developing a method that is more generally applicable than Newton's equation of motion. Naturally, the advantage of Lagrangian mechanics over Newtonian mechanics lies in the fact that it can be used for *generalized coordinate* system. Another advantage is that we do not need to know all the forces acting on the systems or particles within a system. An appropriate choice of generalized coordinates takes care of most of the imposed constraints. This point will become more clear once we look at a few examples.

Hamilton's insight that the same method can be derived from the minimization principle came later. Here, we will not follow Lagrange's development of Lagrangian mechanics in detail. Instead, we will start with Hamilton's principle. Hamilton's principle in its original form states that

> Of all the kinematically possible motions that take a mechanical system from one given configuration to another within a given time interval, the actual motion is the one that minimizes the time integral of the Lagrangian of the system.

In order to dissect this statement, we need to know what Lagrangian is. We will start by defining it, for an unconstrained one dimensional system, as the difference between the kinetic energy, T, and potential energy, V, that is

$$\mathcal{L}(x, \dot{x}) = T(x, \dot{x}) - V(x) \tag{5.3}$$

Once the Lagrangian is defined, the Hamilton's principle becomes,

> Of all the kinematically possible motions that take a mechanical system from one given configuration to another within a given time interval, the actual motion is the one that minimizes the integral
>
> $$S[x] = \int_{t_0}^{t_1} \mathcal{L}(x, \dot{x})dt. \tag{5.4}$$

By applying the Euler-Lagrangian equation(Eq. (5.2)) to minimize the above integral, we get

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0 \tag{5.5}$$

which is referred to as the Lagrange equation. With the kinetic energy and potential energy given by

$$T = \frac{1}{2}m\dot{x}^2 \qquad \text{and} \qquad V = -\int F(x)dx \tag{5.6}$$

Eq. (5.5) reduces to

$$\begin{aligned}\frac{\partial}{\partial x}(T-V) - \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(T-V)\right] &= -\frac{dV}{dt} - \frac{d}{dt}(m\dot{x}) \\ &= F(x) - m\ddot{x} = 0\end{aligned} \tag{5.7}$$

which is identical to the Newton's equation of motion. For a system in higher dimensional space or with multiple particles, we can redefine Lagrangian by calculating the total kinetic energy and potential energy,

$$\begin{aligned}T = \sum_i^n T_i = \sum_i^n \frac{1}{2}m\dot{x}_i^2 \qquad V = V(x_1, x_2, \cdots, x_n) \\ \mathcal{L} = T - V = \mathcal{L}(x_1, x_2, \cdots, x_n, \dot{x}_1, \dot{x}_2, \cdots, \dot{x}_n)\end{aligned} \tag{5.8}$$

One can set Euler-Lagrangian equation for each coordinate and Newtonian equations of motion will be recovered. This seems like a convoluted way of reaching the same conclusion. However, this method becomes extremely powerful in more complicated problems.

5.3 Generalized Coordinates

As we represent motions of N particles in three dimensional space, we needed to construct a configuration space of $3N$-dimensions. But even if the particles are placed in three dimensional space, sometimes their motions can be constrained to some surfaces or lines. In such cases, appropriate coordinate transformations can reduce the number of dimensions required to describe the motion.

Let's first imagine an unconstrained single particle in three dimensional space. Using Newtonian method, we need to set up three equations of motion, one for each coordinate axis.

$$F_i(\mathbf{r}) = m\ddot{x}_i \tag{5.9}$$

where the index $i = 1, 2, 3$ denotes three coordinate axes x, y, and z, respectively.

Now we consider a particle constrained to a circle on the xy-plane. This immediately sets two constrains to the motion

$$x^2 + y^2 = R^2 \qquad \text{and} \qquad z = 0 \tag{5.10}$$

where R is the radius of the ring. One can easily see that the motion is not three dimensional, but how many dimensions should be consider? The answer, surprisingly, is one, not two. The motion certainly takes place in xy-plane which

is a two dimensional surface and it seems obvious that we need two coordinates x and y to describe the motion.

In other words, one of the constraints, $z = 0$, is a solution to the Eq. (5.9) with $i = 3$, and thus simply reduces the three dimensional problem into a two dimensional one. Now the question becomes, how should we apply the other constraint $x^2 + y^2 = R^2$? In principle, $x^2 + y^2 = R^2$ can be used to reduce the second equation of motion from Eq. (5.9) so that we are left with only one equation to solve.

In practice, this turns out to be a near impossible task, because substituting for $y = \sqrt{R^2 - x^2}$ and solving for $F_y = m\ddot{y}$ will give you a headache that will last for days. What we want to do here is to pick a coordinate system that reflects the given constraints more intuitively. That coordinate system, in this case, is the cylindrical coordinate system (3D) or polar coordinate system (2D). By setting $x = r\cos\theta$ and $y = r\sin\theta$, we can replace the constraint as $r = R$. Then we get $x = R\cos\theta$ and $y = R\cos\theta$, which are parametrized with a single variable θ. In essence, we have changed the mechanical problem of a particle moving on a two dimensional xy-plane into a particle moving in a one dimensional θ-space.

More generally speaking, if we start with $3N$ dimensions and there are s number of constraints, we need $n = 3N - s$ dimensions to represent the motion. Depending on the nature of the constraints, the remaining dimensions are not necessarily positions in Cartesian coordinates, or even spatial representations. As long as we can specify coordinate transformations between the $3N$ Cartesian coordinates and these n new coordinates, $\mathbf{q} = (q_1, q_2, \cdots, q_n)$, we can take almost anything. Such n-coordinates are referred to as the *generalized coordinates*. The number of generalized coordinates required, n, is also referred to as degrees of freedom. Generalized coordinates have to meet two conditions:

(i) The generalized coordinates must be independent of each other, that is, there can be no functional form connecting two different coordinates.

(ii) They must fully specify the configuration of the system, that is for any given values of $q_1, q_2, \cdots, q_n$, the position of all the N particles $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N$ must be identifiable. In other words, the position vectors of N particles must be known functions of the n independent generalized coordinates:

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \cdots, q_n) \tag{5.11}$$

It takes some practice to pick appropriate generalized coordinates, but a good choice of the generalized coordinates usually reflects the symmetry of the motion that we consider. In the previous example of the particle on a circular ring, the circular symmetry dictates that we pick polar angle θ as the generalized coordinate.

However, this doesn't help us solve Newtonian equation of motion all that much better. We are still left with a problem of expressing the equation of motion in the generalized coordinate system, which is not trivial. This is where Lagrangian mechanics (Lagrangian equations of motion) comes in. Unlike Newtonian mechanics where the equations are based on vectors, *i.e.* forces, coordinate transformations between the $3N$-dimensional configuration space and the n-dimensional generalized configuration space are extremely cumbersome. However, because Lagrangian mechanics' starting point is a scalar, *i.e.* the

Lagrangian $\mathcal{L}$, coordinate transformation can be implemented in a much more straightforward manner.

5.4 Lagrangian Mechanics

With the coordinate transformations between the Cartesian coordinates and the generalized coordinates available (see Eq. (5.11)), we can reconstruct Lagrangian in the generalized coordinate system. A mechanical motion implies time dependence in $\mathbf{r}_i$'s and this naturally leads to time dependence in $\mathbf{q}$. Thus we can take time derivative of $\mathbf{q}$ and obtain

$$\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n) \tag{5.12}$$

which is called the generalized velocity, since it represents the velocity of the point $\mathbf{q}$ as it moves through the configuration space. One can easily derive the relationship between the real velocity of a particle and the generalized velocity through a simple chain rule for differentiation,

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_1}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \mathbf{r}_n}{\partial q_n}\dot{q}_n \tag{5.13}$$

and kinetic energy of the total system, in terms of the generalized velocity, becomes

$$T = \sum_j^n \sum_k^n a_{jk}(\mathbf{q})\dot{q}_j\dot{q}_k \tag{5.14}$$

where

$$a_{jk}(\mathbf{q}) = \frac{1}{2}\sum_i^N m_i \left(\frac{\partial \mathbf{r}_i}{\partial q_j}\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}\right) \tag{5.15}$$

The kinetic energy T is then a function of $\mathbf{q} = (q_1, \cdots, q_n)$ and $\dot{\mathbf{q}} = (\dot{q}_1, \cdots, \dot{q}_n)$. Similarly, the potential energy V can be rewritten as a function of $\mathbf{q}$. Then the Lagrangian becomes a function of the generalized coordinates and generalized velocities:

$$\mathcal{L} = \mathcal{L}(q_1, \cdots, q_n, \dot{q}_1, \cdots, \dot{q}_n) \tag{5.16}$$

In applying Hamilton's principle to mechanics, there is no reason to believe that it should work in real coordinates only. In other words, we can apply the calculus of variations using Lagrangian expressed in the generalized coordinates, and we are then left with n number of Lagrange equations to solve

$$\frac{\partial \mathcal{L}}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) = 0 \tag{5.17}$$

where $i = 1, 2, \cdots, n$.

By rewriting the equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} = \mathcal{Q}_j \tag{5.18}$$

we can see that there is a term, Q_j, that is defined in the generalized coordinate, corresponding to the concept of force in the real space. For this reason, Q_j is

known as the generalized force. By applying the chain rule in calculating this derivative of potential with respect to the generalized coordinates, it is easy to prove that the generalized force and the real space force has the relationship:

$$\mathcal{Q}_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{5.19}$$

where $\mathbf{F}_i$ is the specified force acting on the i-th particle.

So far, we have only considered a constraint that can be expressed in the form of $\mathbf{r}_i = \mathbf{r}_i(\mathbf{q})$. Such constraints are referred to as geometric constraints, because the coordinate transformation between the real coordinates and generalized coordinates is purely geometric, that is there is no explicit time dependence or velocity dependence in the coordinate transformation. Such a system is said to be holonomic. Establishing Lagrangian mechanics for a non-holonomic system in general is outside the scope of this course and we will not treat the problem here.

However, time dependent constraint can be relatively easily integrated into the Lagrange equations. Also, there are some cases where the generalized force can be expressed in the form

$$\mathcal{Q}_j = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right) - \frac{\partial V}{\partial q_j} \tag{5.20}$$

for some function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$. Then the function $V(\mathbf{q}, \dot{\mathbf{q}}, t)$ is called the velocity dependent potential of the system and Lagrangian is given by

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{5.21}$$

and one can still use Lagrange equation to solve mechanics problems. Note that the velocity dependent potential produces non-conservative force as the resulting force is not strictly position dependent.

5.5 D'Alembert's Principle

It seems pretty obvious, by now, that with the help of Hamilton's principle and the definition of Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - V(\mathbf{q}, \dot{\mathbf{q}}, t)$, one can easily come up with a mechanical equation in generalized coordinates whose solution can be transformed later into a real space solution. However, there is a nagging question regarding the definition of Lagrangian, that is "how did someone come up with such a physical quantity that produces the Lagrangian equation?" The answer to this question lies, as stated earlier, in the fact that the Hamilton's principle came after Lagrange came up with his definition of Lagrangian. For Lagrange, the starting point was virtual work and D'Alembert's principle.

Suppose a system with N particles where an individual particle is subject to a net force,

$$\mathbf{F}_i = \mathbf{F}_i^S + \mathbf{F}_i^C = m_i \dot{\mathbf{v}}_i \tag{5.22}$$

where $\mathbf{F}_i^S$ denotes the specified force and $\mathbf{F}_i^C$ denotes the constraint force. In other words, we know explicitly how the $\mathbf{F}_i^S$'s act on each particle, but the effect of $\mathbf{F}_i^C$'s is not known to us, except for the fact that the constraint force acts as a boundary condition. For most cases, the direction of such constraints

are orthogonal to the directions of particle's allowed motion. The effect of the constraint can be thus written as

$$\sum_i^N \mathbf{F}_i^C \cdot \mathbf{v}_i = 0 \tag{5.23}$$

In other words, we can rewrite the equation of motion for the total system as

$$\sum_i^N m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i = \sum_i^N \mathbf{F}_i^S \cdot \mathbf{v}_i + \sum_i^N \mathbf{F}_i^C \cdot \mathbf{v}_i = \sum_i^N \mathbf{F}_i^S \cdot \mathbf{v}_i \tag{5.24}$$

where the constraint force is no longer explicit.

However, one can consider a virtual path that is also subject to the same constraint force, and by denoting the velocity along that virtual path as $\mathbf{v}_i^*$, we can get the identical relation

$$\sum_i^N m_i \dot{\mathbf{v}}_i \cdot \mathbf{v}_i^* = \sum_i^N \mathbf{F}_i^S \cdot \mathbf{v}_i^* \tag{5.25}$$

which is known as the D'Alembert's principle.

For a holonomic system with constraint force doing no virtual work, there must be a generalized coordinate system $\mathbf{q}$ that links the real space position of the particles $\mathbf{r}_i$'s to $\mathbf{q}$. With such generalized coordinate system, we can construct a virtual motion $\mathbf{v}_i^*$

$$\mathbf{v}_i^* = \frac{\partial \mathbf{r}}{\partial q_1} \tag{5.26}$$

that corresponds to the motion generated by generalized velocities

$$\dot{q}_1 = 1, \dot{q}_2 = 0, \cdots, \dot{q}_n = 0 \tag{5.27}$$

From d'Alembert's principle, we get

$$\sum_i^N m_i \dot{\mathbf{v}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_1} = \sum_i^N \mathbf{F}_i^S \cdot \frac{\partial \mathbf{r}_i}{\partial q_1} \tag{5.28}$$

We can consider different virtual motion generated by $\dot{q}_1 = \dot{q}_2 = \cdots = \dot{q}_n = 0$ for all the generalized coordinates except for $\dot{q}_j = 1$. For all the possible values of j, we get n equations

$$\sum_i^N m_i \dot{\mathbf{v}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i^N \mathbf{F}_i^S \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{5.29}$$

The left hand side of the equation can be rewritten in terms of q_j's

$$\sum_i^N m_i \dot{\mathbf{v}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = \sum_i^N \mathbf{F}_i^S \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \tag{5.30}$$

which is identical to the Lagrange equation given by Eq. (5.18). Now one can work backwards what we treated in the previous sections and define Lagrangian as $T - V$ and also come up with Hamilton's principle.

5.6 Conjugate Variables

Up to this point, we have treated how an appropriate choice of generalized coordinates can simplify solving mechanical problems. According to such choice, we have constructed generalized velocities and generalized forces. It is only natural to suspect that there must be generalized momenta. In Lagrangian mechanics, the generalized momentum corresponding to the coordinate q_i is defined as

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{5.31}$$

It is also called the momentum conjugate to q_i. In this case, the generalized coordinate and the generalized momentum are said to be conjugate variables. The Lagrangian equation can be rewritten in terms of the generalized momentum

$$\frac{\partial \mathcal{L}}{\partial q_i} = \dot{p}_i \tag{5.32}$$

From this, one can easily see that the generalized momentum, p_i, is conserved if the corresponding coordinate, q_i is cyclic, *i.e.* absent from the Lagrangian.

In general, the Lagrangian is a function of $\mathbf{q}$, $\dot{\mathbf{q}}$ and t. The case where $\dot{q}_i$ is absent from Lagrangian is a special case of q_i being cyclic with zero generalized momentum. An interesting point here to consider is what happens if the time t is cyclic? Is there a conjugate momentum to time? To answer this question, we will move on to the next topic, the Hamiltonian mechanics.

Chapter 6

Hamiltonian Mechanics

We now turn our attention to Hamilton's formalization of mechanics. It should be made clear from the very beginning that in terms of solving problems in classical mechanics, Hamiltonian mechanics provides almost no advantage over Lagrangian mechanics. In fact, in most cases, things appear to be more difficult to solve with Hamiltonian mechanics. The benefits of Hamiltonian mechanics lie in interpretation of mechanical motion and as a bridge towards statistical mechanics and quantum mechanics.

6.1 Legendre Transformation: From Lagrangian to Hamiltonian

In the previous chapter, we defined generalized momentum p_i as the conjugate momentum of the generalized coordinate, q_i, as

$$p_j = \frac{\partial}{\partial \dot{q}_j} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{6.1}$$

and this yields the Lagrangian equation of motion into the following form

$$\dot{p}_j = \frac{\partial}{\partial q_j} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t). \tag{6.2}$$

Here we have a differential equation for p_j expressed in terms of variables $\mathbf{q}$, $\dot{\mathbf{q}}$ and t. To have a self-consistent form of the differential equation, it would be desirable to formulate the equation in terms of $\mathbf{p}$ instead of $\dot{\mathbf{q}}$. This can be achieved by performing a functional transformation known as the Legendre transformation. For anyone that is interested in the mathematics of the derivation, detailed calculation is given in the appendix. By performing the transformation, we can construct a function $\mathcal{H}(\mathbf{q}, \mathbf{p}, t)$ as below

$$\mathcal{H}(\mathbf{q}, \mathbf{p}, t) = \dot{\mathbf{q}} \cdot \mathbf{p} - \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t). \tag{6.3}$$

Here, the variables that do not participate in transformation, *i.e.* $\mathbf{q}$ and t, are referred to as passive variables, and naturally the variables that do participate in transformation, *i.e.* $\dot{\mathbf{q}}$ and $\mathbf{p}$, are called the active variable.

At first glance, it is not obvious that the new function $\mathcal{H}$ is a function of $\mathbf{q}$, $\mathbf{p}$ and t. Rather, it seems like a function of all four variables $\mathbf{q}$, $\dot{\mathbf{q}}$, t *and* $\mathbf{p}$. To see this, we will take a differential of $\mathcal{H}$:

$$\begin{aligned} d\mathcal{H} &= \sum_{i=1}^{n}(\dot{q}_i dp_i + p_i d\dot{q}_i) - \sum_{i=1}^{n}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}d\dot{q}_i - \sum_{i=1}^{n}\frac{\partial \mathcal{L}}{\partial q_i}dq_i - \frac{\partial \mathcal{L}}{\partial t}dt \\ &= \sum_{i=1}^{n}\dot{q}_i dp_i + \sum_{i=1}^{n}\left(p_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)d\dot{q}_i - \sum_{i=1}^{n}\frac{\partial \mathcal{L}}{\partial q_i}dq_i - \frac{\partial \mathcal{L}}{\partial t}dt \end{aligned} \tag{6.4}$$

From the definition of the generalized momentum, the second term in the second line of the above equation is zero, that is, the differential of $\mathcal{H}$ does not contain any term with $d\dot{\mathbf{q}}$. In other words, the function $\mathcal{H}$ is not a function of $\dot{\mathbf{q}}$s, but only of $\mathbf{q}$, $\mathbf{p}$ and t. This function, $\mathcal{H}$, is called the Hamiltonian function or just Hamiltonian.

6.2 Hamilton's Equations

For two functions $\mathcal{H}(\mathbf{q}, \dot{\mathbf{q}}, t)$ and $\mathcal{H}(\mathbf{q}, \mathbf{p}, t)$ that are Legendre transformation of each other, following relations hold:

$$\frac{\partial \mathcal{L}}{\partial q_j} = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{6.5}$$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \tag{6.6}$$

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j} \tag{6.7}$$

with $\mathbf{p}$ being a passive variable and $\dot{\mathbf{q}}$ and $\mathbf{p}$ as active variables. Again proof of these relations are presented in appendix and we will simply use this relation here. Combining Eq. 6.5 and Eq. 6.6 with Lagrangian equation of motion, Eq. 5.17, we can show that

$$\dot{p}_j = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) = \frac{\partial \mathcal{L}}{\partial q_j} = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{6.8}$$

Inspecting Eq. 6.7 and Eq. 6.8, one can see that they form a set of $2n$ first order differential equations that specifies all the relationship required to solve for $\mathbf{q}$ and $\mathbf{p}$. These are the Hamiltonian equations of motion or simply Hamilton's equations:

$$\dot{q}_j = \frac{\partial \mathcal{H}}{\partial p_j} \qquad\qquad \dot{p}_j = -\frac{\partial \mathcal{H}}{\partial q_j} \tag{6.9}$$

In other words, we have just swapped out Lagrange equations, which are n second order differential equations, with Hamilton's equation, $2n$ first order differential equations.

Note that there is a nice symmetry to the equations. It will become more obvious with a different notation for the variables, ξ_k:

$$\xi_k = \begin{cases} q_k & \text{for} \quad k = 1, 2, \cdots, n \\ p_{k-n} & \text{for} \quad k = n+1, \cdots, 2n \end{cases} \tag{6.10}$$

Then the Hamilton's equations become

$$\begin{aligned} \dot{\xi}_k &= \frac{\partial \mathcal{H}(\boldsymbol{\xi}, t)}{\partial \xi_{k+n}} \qquad \text{for} \quad k = 1, 2, \cdots, n \\ \dot{\xi}_k &= -\frac{\partial \mathcal{H}(\boldsymbol{\xi}, t)}{\partial \xi_{k-n}} \qquad \text{for} \quad k = n+1, \cdots, 2n \end{aligned} \tag{6.11}$$

Since the RHS of the above equations result in some functions of $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_{2n})$ and t, for all values of k running from 1 to $2n$, the equations can be rewritten as:

$$\dot{\xi}_k = f_k(\xi_1, \xi_2, \cdots, \xi_{2n}, t) \tag{6.12}$$

In this regard, the generalized coordinates $\mathbf{q}$ and the generalized momentum $\mathbf{p}$ are indistinguishable. In other words, q_js and p_js are independent variables that are on equal footings.

Aside from the beautiful simplicity in expressing the equations of motion as Eq. 6.12, it is not obvious at all why one would formulate the equations in this way. As a matter of fact, unless the Hamiltonian is given a priori, the process described above is not any less painful than the Lagrangian mechanics. Setting up Hamilton's equations of motion requires constructing Hamiltonian from Lagrangian, and there is a step that makes finding Hamilton's equations from Hamiltonian practically meaningless.

To illustrate this point, we turn to our favorite example once again, one dimensional simple harmonic oscillator. We start from the Lagrangian

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \tag{6.13}$$

and the generalized momentum is then

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m \dot{x} \tag{6.14}$$

The velocity, $\dot{x}$, is then expressed in terms of the momentum, p,

$$\dot{x} = \frac{p}{m} \tag{6.15}$$

and this lets us formulate Hamiltonian

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{2m} + \frac{1}{2} k x^2. \tag{6.16}$$

Hamiltonian has to be expressed in terms of $\mathbf{q}$, $\mathbf{p}$ and t only, and the generalized velocity $\dot{\mathbf{q}}$ must be expressed as a function of the generalized momentum $\mathbf{p}$. This is done, in the above example, by converting Eq. 6.14 into Eq. 6.15 and then Eq. 6.15 being plugged into Eq. 6.16. Once the Hamiltonian is obtained, we can find the Hamilton's equations:

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \tag{6.17}$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -kx \tag{6.18}$$

However, notice that the first of the two equations, Eq. 6.17, is nothing but Eq. 6.15. The fact that two equations, Eq. 6.15 and Eq. 6.17, are identical is

no surprise, as it is a salient feature of Legendre transformation. However, a part of the Hamilton's equations are produced in the process of constructing Hamiltonian makes the whole process somewhat circular.

An attempt to solve the Hamilton's equations, Eq. 6.17 and Eq. 6.18, inflicts more disappointment. Because two Hamilton's equations are coupled, that is $\dot{x}$ is a function of p and $\dot{p}$ is a function of x, to solve these equations, one ends up differentiating Eq. 6.17 with respect to time and plugging it into Eq. 6.18. The end result yields

$$\ddot{x} + \frac{k}{m}x = 0 \tag{6.19}$$

which is identical to the Lagrangian equation (and Newtonian equation). So what is the point of all this practice other than causing you a massive headache? To find (one of) the answer(s), we'll take a trip into the *phase space*.

6.3 Configuration Space and Phase Space

In formulating the Lagrangian mechanics, we emphasized the importance of finding n independent generalized coordinates, $\mathbf{q} = (q_1, q_2, \cdots, q_n)$ forming the configuration space. The Lagrange equation yields time dependent solutions for the $q_j(t)$'s. Since we can trace out the motions of a mechanical system of interest in real space with an appropriate coordinate transformations, $\mathbf{r}_i(t) = \mathbf{r}_i(q_1(t), q_2(t), \cdots, q_n(t))$, for all N particles, any mechanical motion can be represented by a point $\mathbf{q}(t) = (q_1(t), q_2(t), \cdots, q_n(t))$ moving through the configuration space.

As powerful as this method seems, this is not an effective method of visually representing a mechanical motion, as this involves time evolution of a point along the phase space. We have seen in the previous chapter that a trajectory in the configuration space, that is a graphical representation of the motion obtained by compressing the time evolution and projecting all the motion on to the configuration space, is not a unique identification of motion.

It is quite obvious why this is so. The Lagrangian equations of motion are second order differential equations. To exactly solve a second order differential equation, we need two initial conditions, not one to obtain the exact solution. In Newtonian mechanics, these two initial conditions were the position and velocity. A point in configuration space specifies the initial positions but not initial velocities.

This shortcoming is averted in the Hamiltonian formulation as the mechanical motion is represented in a $2n$-dimensional space spanned by both $\mathbf{q}$ and $\mathbf{p}$. As we said in the previous section, in Hamiltonian mechanics, $\mathbf{q}$ and $\mathbf{p}$ are independent variables that need to be solved from the $2n$ first order differential equations simultaneously. Therefore, at any given moment, the solution to the equation is expressed as $\mathbf{q}(t)$ and $\mathbf{p}(t)$. A graphical representation, thus requires a point in $\mathbf{q}$-$\mathbf{p}$ space, a $2n$-dimensional vector space also known as the *phase space*.

The phase space by nature is $2n$-dimensional and identifying any point inside the phase at a given moment provides us with $2n$ initial conditions. Therefore, a trajectory in phase space has no ambiguity unlike that of a configuration space.

To make this point clear, let us look at an example of simple harmonic oscillators in one dimension. Imagine pulling on a particle with mass m attached

to a spring with spring constant α. We pull the particle by a displacement of A, hold it still and then let go of it. Here, there is only one generalized coordinate x, which is identical to the real coordinate of the particle. Thus the configuration space is simply one dimensional space represented by x. A trajectory in the configuration space is a line between $x = -A$ and $x = A$ as shown in Fig. **??**.

Let's consider another particle with mass m attached to a spring with spring constant 4α this time. If the initial displacement is again $x = A$, the trajectory in the configuration space is identical to the one from above (See Fig. **??**). Put differently, without the explicit time evolution along the trajectory, the trajectory itself does not allow us to identify the motion in configuration space.

We will now look at these two cases in the phase space. For the first case with spring constant of α, any loss of potential energy, $\alpha(A^2 - x^2)/2$, is converted into kinetic energy, $p^2/2m$, and thus we have the relation

$$\frac{1}{2}\alpha x^2 + \frac{p^2}{2m} = \frac{1}{2}\alpha A^2 \tag{6.20}$$

This is an equation of ellipse in the x-p phase space (See Fig. **??**). The second spring with four times as big spring constant yields the phase space trajectory along

$$2\alpha x^2 + \frac{p^2}{2m} = 2\alpha A^2 \tag{6.21}$$

which is a totally different ellipse than the first one. When trajectories such as these ellipses are presented in the phase space, each of them uniquely defines the motion and any time evolution of a point on the trajectory can be exactly estimated.

However, there is another subtle but possibly more important point in phase space. For a given set of equations of motion, there is only one trajectory that passes through any given point in the phase space. In other words, trajectories in phase space do not overlap or cross one another. This is equivalent to saying that when the initial conditions are fully specified, there is one and only one exact solution to the equations of motion.

We will take an example of a particle with mass m that can move only in one dimension, say x-axis. The particle is under a constant force F along the positive x direction. The general solution to this problem in Hamiltonian mechanics is

$$x(t) = \frac{1}{2}\frac{F}{m}t^2 + \frac{p_0}{m}t + x_0 \tag{6.22}$$

where x_0 and p_0 are initial position and momentum of this particle, respectively. The proof of this is left for the readers. Also, from the Hamilton's equations, one can easily show that

$$p(t) = p_0 + Ft \tag{6.23}$$

Combining the two solutions, we have the phase space trajectory:

$$x = \frac{p^2}{2mF} - \frac{p_0^2}{2mF} + x_0 \tag{6.24}$$

which gives us an infinite set of parabolas depending on the values of x_0 and p_0, a few of which are drawn in Fig. **??**. The key feature here is that for any point inside the phase space, there is only one trajectory that passes through that point. The point is permanently attached to that trajectory and flows along it in a specific direction.

6.4 Hamiltonian and Energy

To figure out another important aspect of Hamiltonian mechanics, let us take a time derivative of the Lagrangian in its most general form:

$$\frac{d}{dt}\mathcal{L}(\mathbf{q},\dot{\mathbf{q}},t) = \sum_{i=1}^{n}\left(\frac{\partial\mathcal{L}}{\partial q_i}\dot{q}_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\ddot{q}_i\right) + \frac{\partial\mathcal{L}}{\partial t} \tag{6.25}$$

By now, you should be able to identify immediately that

$$\frac{\partial\mathcal{L}}{\partial q_i} = \dot{p}_i \qquad\qquad \frac{\partial\mathcal{L}}{\partial\dot{q}_i} = p_i \tag{6.26}$$

and Eq. 6.25 becomes

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{n}(\dot{p}_i\dot{q}_i + p_i\ddot{q}_i) + \frac{\partial\mathcal{L}}{\partial t} = \frac{d}{dt}\left(\sum_{i=1}^{n} p_i\dot{q}_i\right) + \frac{\partial\mathcal{L}}{\partial t} \tag{6.27}$$

From this, we can see that the time derivative of the Lagrangian is generally not constant even if the Lagrangian has no explicit time dependence, *i.e.* $\frac{\partial\mathcal{L}}{\partial t} = 0$. From Eq. 6.3, we can see that the first term in the RHS of the above equation is $\mathcal{L} + \mathcal{H}$:

$$\frac{d\mathcal{L}}{dt} = \frac{d}{dt}(\mathcal{L}+\mathcal{H}) + \frac{\partial\mathcal{L}}{\partial t}. \tag{6.28}$$

Therefore, we have

$$\frac{d\mathcal{H}}{dt} = -\frac{\partial\mathcal{L}}{\partial t}. \tag{6.29}$$

that is, if the Lagrangian is not an explicit function time, the Hamiltonian is conserved. A fancy way of saying is that if the system is time-translation invariant, then the Hamiltonian is conserved. Going back to the last question we had at the end of the previous chapter, we can see that the Hamiltonian is the conjugate momentum of time.

In fact, it is quite straightforward to show that, for very simple conservative systems such as a harmonic oscillator, or a particle under the influence of a constant force, that the Hamiltonian is sum of the kinetic energy and the potential energy, *i.e.* total mechanical energy. We leave it as an exercise for readers. However, the Hamiltonian is a more general form of energy that the total mechanical energy given by $E = T + V$ and on this ground, we equate Hamiltonian with the total energy of the system.

The identification of Hamiltonian as the total energy allows us to approach mechanics of conservative system without actually solving the equation of motion. We go back to the single particle under constant force in one dimension. Since the force is constant throughout space, the system is conservative. Thus we can write down the Hamiltonian as

$$\mathcal{H} = T + V = \frac{p^2}{2m} - Fx = \text{constant}(E). \tag{6.30}$$

If we solve the above equation for x, we get

$$x = \frac{p^2}{2mF} - \frac{E}{F}. \tag{6.31}$$

One can immediately notice the similarity between the above equation and the Eq. 6.24. Upon a close inspection, we can see that they are, in fact, identical. For the given initial position and momentum of x_0 and p_0, the total energy of the system is $E = \frac{p_0^2}{2m} - Fx_0$, and substituting this for E in Eq. 6.31, we get exactly the same equation as Eq. 6.24.

This raises two interesting points for the case where the Lagrangian has no explicit time dependence. First is that the Hamiltonian also has no explicit time dependence, and we have $\mathcal{H}(\mathbf{q}, \mathbf{p}) = E$. This is an equation for a $(2n-1)$-dimensional surface in the $2n$-dimensional phase space. We can therefore obtain the surface along which the motion takes place in the phase space without having to solve any differential equations. Exact time evolution is not immediately accessible, but a geometrical solution in the phase space is directly handed to us.

The second point is that the two initial conditions x_0 and p_0 are rolled up in the total energy E. This is a combined effect of uniqueness of phase space trajectory (or surface) and time-translation invariance. As we emphasized earlier, for any given initial condition, the phase space trajectory is unique and a point in phase space cannot escape from that trajectory. Also, the geometric solution, *i.e.* the trajectory or surface, in the phase space is not time stamped, and it represents the entire history of the motion from negative infinite time to positive infinite time. If the system has time-translation invariance, that is, if the system is unaffected by sliding up and down along the time axis, it means we can slide identification of certain position, x_0, and momentum, p_0, at a given moment, t_0, loses its meaning. Without actually following the time evolution, we can slide in time from t_0 to $t_0 + \Delta t$ by an arbitrary amount Δt and set that moment as our initial moment. It will shift the initial conditions to $x_0 + \Delta x$ and $p_0 + \Delta p$, yet remains on the same trajectory on the phase space. Thus, the only thing that is relevant in identifying the solution is the phase space trajectory determined by the total energy E and not the initial conditions x_0 and p_0.

Chapter 7

Central Force Motion

From the historical point, central force motion is one of the most important problems in classical mechanics. It was people's desire to understand planetary motions and stellar objects that gave rise to much of Newton's finest works, and it turns out that the planets and stellar objects are governed by gravity, one of the two most representative cases of the central forces, along with Coulomb force in electrodynamics.

Central force is a force with spherical symmetry that only depends on the distance from the center of the force field. In other words, a force field $\mathbf{F}(\mathbf{r})$ is said to be a central force field with center O if it has the form

$$\mathbf{F}(\mathbf{r}) = F(r)\hat{\mathbf{r}} \tag{7.1}$$

where $r = |\mathbf{r}|$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. Gravity falls in this category as the force between two massive objects with masses m and M is given by the relation

$$F(r) = -\frac{GmM}{r^2} \tag{7.2}$$

Note that central force is a function of position and thus is a conservative force with potential, $V(r)$, satisfying

$$F(r) = -\frac{dV(r)}{dr} \tag{7.3}$$

Also, due to the symmetry of the force field, an object's motion under central force field is always planar, that is, the object is bound in a plane formed by the center of the force field, the position and the velocity vector of the object under the influence of the central force. Thus two dimensional polar coordinate is sufficient to describe the motion in its full detail. In the following sections, we have performed all the calculations with the mass of object included. Note that in the textbook by Gregory, they have performed all the calculations for unit mass. We leave it to you to try and convert the equations between two different forms. It is something you should be able to do.

7.1 Conservation Laws in Central Force Field

The central force conserves the system's angular momentum. This is because there is no net torque acting on the particle as the force is always along the

direction of the moment arm,

$$\mathbf{K} = \mathbf{r} \times \mathbf{F} = F(r)\mathbf{r} \times \hat{\mathbf{r}} = \mathbf{0} \tag{7.4}$$

This can be also seen from the Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \tag{7.5}$$

which is cyclic about θ, and the angular momentum p_θ is conserved,

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = \text{constant} = l \tag{7.6}$$

The angular momentum conservation can be rewritten as

$$\dot{\theta} = \frac{l}{mr^2} \tag{7.7}$$

$$\theta(t) = \int_0^t \frac{l}{m[r(\tau)]^2} d\tau, \tag{7.8}$$

which means that once the time dependence of $r(t)$ is known, the time dependence of $\theta(t)$ can be easily found out.

The second Lagrange equation is

$$F(r) + mr\dot{\theta}^2 - m\ddot{r} = F(r) + \frac{l^2}{mr^3} - m\ddot{r} = 0 \tag{7.9}$$

This is an ordinary differential equation for the radial distance $r(t)$ which is generally not solvable.

Instead of solving the differential equation, we can obtain qualitatively description of the motion based on another feature of the Lagrangian. The Lagrangian is not an explicit function of t, and thus the energy function h must be conserved.

$$h = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = \text{constant} = E \tag{7.10}$$

Using Eq. (7.7), we can rewrite the above equation as

$$\frac{1}{2}m\dot{r}^2 + V(r) + \frac{l^2}{2mr^2} = E \tag{7.11}$$

In fact, this is identical to Eq. (7.9) as taking the time derivative of the above equation yields Eq. (7.9). One important feature of the above equation is that the angular velocity component of the kinetic energy is reformulated as a function of r,

$$\frac{1}{2}mr^2\dot{\theta}^2 = \frac{l^2}{2mr^2} \tag{7.12}$$

and this allows us to define the effective potential

$$V_{eff}(x) = V(x) + \frac{l^2}{2mx^2} \tag{7.13}$$

We can then treat the system as a one dimensional particle moving with kinetic energy $T = \frac{1}{2}m\dot{x}^2$ under the potential energy $V_{eff}(x)$:

$$\frac{1}{2}m\dot{x}^2 + V_{\text{eff}}(x) = E \tag{7.14}$$

with $x > 0$. Even if we cannot solve the equation exactly, one can easily define the range of motion in r by plotting the effective potential $V_{\text{eff}}(x)$ and E.

7.2 The Path Equation

Solving the Eqs. (7.8) and (7.9) can give us a full picture of the time dependent motion. However, with $r(t)$ and $\theta(t)$ given, we can use them to get parametric equation $r(\theta)$, which is typically dubbed the orbit of the system.

Instead of having to find $r(t)$ and $\theta(t)$ to find the orbit, one can combine Eq. (7.9) and (7.7) to eliminate time. By introducing a new variable $u = 1/r$,

$$\ddot{r} = -\frac{l^2u^2}{m^2}\frac{d^2u}{d\theta^2} \tag{7.15}$$

$$r\dot{\theta}^2 = \frac{l^2u^3}{m^2} \tag{7.16}$$

and we can rewrite Eq. (7.9) as

$$\frac{d^2u}{d\theta^2} + u + \frac{mF(1/u)}{l^2u^2} = 0 \tag{7.17}$$

This is the path equation. Only for inverse square law and inverse cube laws, the path equation becomes a linear equation. Otherwise, this is a non-linear equation. It is rather fortunate that the gravity is an inverse square law, which can be solved analytically.

We will, in fact, take a look at the attractive inverse square law where $F(r) = -\gamma/r^2 = -\gamma u^2$. Then the path equation is a linear equation of the form

$$\frac{d^2u}{d\theta^2} + u - \frac{m\gamma}{l^2} = 0 \tag{7.18}$$

The general solution of the equation is

$$\frac{1}{r} = A\cos(\theta - \alpha) + \frac{m\gamma}{l^2} = \frac{m\gamma}{l^2}\left(1 + e\cos(\theta - \alpha)\right) \tag{7.19}$$

Since we are interested in the path of orbital motion, we can set α to be zero and, without losing generality, rewrite the equation

$$\frac{1}{r} = \frac{m\gamma}{l^2}\left(1 + e\cos\theta\right) \tag{7.20}$$

The eccentricity $e = \frac{Al^2}{m\gamma}$ is related to the initial condition and if $e = 0$, the path is a circle. If $e < 1$, then the path is an ellipse. For $e = 1$ and $e > 1$, the results are parabola and hyperbola respectively.

Also, by comparing the Eq. (7.20) to the standard polar equation of a conic

$$\frac{1}{r} = \frac{a}{b^2}\left(1 + e\cos\theta\right) \tag{7.21}$$

we get the relation between the angular momentum l and the conic parameters a and b,

$$l^2 = m\gamma\frac{b^2}{a} \tag{7.22}$$

This relation is known as the L-formula. The total energy E and the conic parameter a can be also shown to have the following relation, E-formula

$$\begin{array}{ll} \text{Ellipse :} & E = -\frac{\gamma}{2a} < 0 \\ \text{Parabola :} & E = 0 \\ \text{Hyperbola :} & E = \frac{\gamma}{2a} > 0 \end{array} \tag{7.23}$$

For a particle moving under the central force field obeying the repulsive inverse square law, the force is given by $F(r) = \gamma/r^2$, and the path equation is correspondingly,

$$\frac{d^2u}{d\theta^2} + u + \frac{m\gamma}{l^2} = 0 \tag{7.24}$$

whose solution can only be a parabola or hyperbola,

$$\frac{1}{r} = \frac{m\gamma}{l^2}\left(-1 + e\cos\theta\right). \tag{7.25}$$

Chapter 8

System of Multiparticles

Unless you've been living in a cave all by yourself until you came to KAIST, I'm sure you've all seen people high jump at some point in your life. Now, when you see high jumping events, there's that one question you can't get shaken off your head. 'Why in the hell are these people jumping over a bar in that awkward posture, twisting your body up in the air, facing backwards not being able to look at where the bar is, and finally flipping backwards?' The secret under this rather bizarre collective behavior of a group of seemingly normal people in any other regards lies within the center of mass. We will get back to high jumping in a minute, and in the mean time let's talk about center of mass.

So far, we've been talking about a single particle with its mass all concentrated at a single point. But most of matters in real life are not point-like and composed of many particles. How are we supposed to deal with such systems? One rather intuitive (or counter-intuitive depending on your intuition) way is to find a single-particle-like representation of systems of particles. And that representation is what we call center of mass.

8.1 Weighted Average

Let's suppose you've taken five classes last semester: differential equations (4), classical mechanics (3), electromagnetism (3), physics lab (2), and English writing (2). The numbers in the parentheses are number of credits for each class. Imagine you've got B, A, A, B, and C in each of these five classes. What would be your GPA for the semester? No one in the right mind would calculate your GPA like $(3.0 + 4.0 + 4.0 + 3.0 + 2.0)/5 = 3.2$. Instead, what you do is the following: $(3.0\times4+4.0\times3+4.0\times4+3.0\times2+2.0\times2)/(4+3+3+2+2) = 3.29$. This is a way of emphasizing classes that are more *important*, and the process is known as calculating weighted-average.

When you have a collection of numbers x_i's with each of their weight equal to w_i, the weighted average can be calculated following the equation,

$$\bar{x} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}. \tag{8.1}$$

This gives us an average value of a set of numbers after taking into account the *weight* or *importance* of these values. By doing so, we can get a single-

valued representation of a set of numbers. When your GPA is 3.29 and someone else's GPA is 3.01, we can safely assume that you've been a better student than the other person without knowing the grades of every single class you've taken. Of course, the person with lower GPA could have gotten a better grade than you in some classes, but overall you've done better. It is a way of compressing information.

8.2 Center of Mass

We can think of your grade as pure numbers, but we can also present it on a number line. It gives us a visual and spatial representation, and your grade becomes a position along a one-dimensional number line. See Fig. 9.1. Such representation is helpful in understanding what center of mass is.

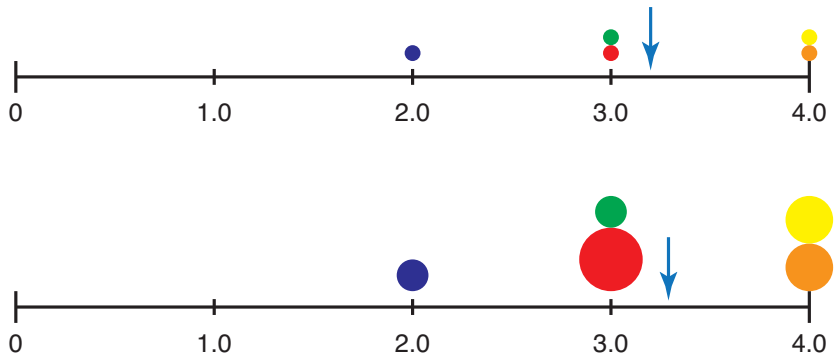

Figure 8.1:

Now, instead of taking an average of your grades, let's consider a system of particles. For now let's just focus on the position of the particles along the x-axis. There are n particles with each particle having mass m_α, and position x_α where α denotes each particle, *i.e.* $\alpha = 1$ through n. Instead of having to specify each particles position, is there a good way to compress this information? Can we treat this system of particles with a single number? If we want to treat a collection of particles as a single entity, the sensible thing to do is think of a particle with the total mass $M = \sum_{\alpha=1}^{n} m_\alpha$.

What about the position of the particle. Is there a single point representation of all these particles? The answer is yes and that is quite literally in this case the *weighted*-average position of all the particles. In other words, we can think of this system of many particles as a single particle with total mass M positioned at

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i. \tag{8.2}$$

We can do the same for y- and z-axes and write it in vector representation.

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{n} m_i \mathbf{r}_i. \tag{8.3}$$

This can further be generalized for a system with total mass M and a continuous mass distribution, *i.e.* a system with mass density, $\rho(\mathbf{r})$, at the point $\mathbf{r}$. The

summation then becomes an integral and the center of mass is

$$\mathbf{R} = \frac{\int \rho(\mathbf{r})\mathbf{r}dV}{\int \rho(\mathbf{r})dV} = \frac{1}{M}\int \rho(\mathbf{r})\,\mathbf{r}\,dV. \tag{8.4}$$

We have thus far calculated the center of mass. Now the natural question that follows is "how is this useful"? We will deal with this question in the next few following sections.

8.3 Linear Momentum

Let's first think about the linear momentum of a system of particles. Each particle in the system would have its own momentum, and we will denote the momentum of the i-th particle as $\mathbf{p}_i = m_i\dot{\mathbf{r}}_i$. If we are to calculate the total momentum of the system with all n-particles in it, we can just sum over i and we get

$$\mathbf{P} = \sum_i \mathbf{p}_i = \sum_i m_i\dot{\mathbf{r}}_i = \frac{d}{dt}\sum_i m_i\mathbf{r}_i = \frac{d}{dt}M\mathbf{R} = M\dot{\mathbf{R}}, \tag{8.5}$$

which is to say:
I. *The linear momentum of the system is the same as if a single particle of mass M were located at the position of the center of mass and moving in the manner the center of mass moves.*

What about the time derivative of the momentum? Before we answer this question, let's actually impose a couple of assumptions. When we have a collection of particles, it is entirely possible that there is some type of force acting on one particle by another. For example, if we have a bunch of electrons, there will be Coulomb force acting between each pair of electrons. This is what we call internal force and we will assume the following about the internal force:

> The internal force acting on a particle i by another particle j is equal in size and opposite in direction to the force acting on the particle j by the particle i. That is $\mathbf{G}_{ij} = -\mathbf{G}_{ji}$, where $\mathbf{G}_{ij}$ denotes the force acting on i by j. This is in fact Newton's Third Law.

With these assumptions in mind, we now turn to external force. It is conceivable that there is a force acting on a particle that is not exerted by other particles that constitute the system. Any force acting on a particle i that is not coming from within the system is what we call external force. We will denote the external force as $\mathbf{F}_i$. In such a case, the total amount of force acting on the particle i, $\mathbf{F}_i^{\text{net}}$ is $\mathbf{F}_i^{\text{net}} = \mathbf{F}_i + \sum_j \mathbf{G}_{ij}$. Here, the internal force on the i-th particle, $\sum_j \mathbf{G}_{ij}$, is summed over all the other particles $j \neq i$'s in the system. (Alternatively, we can set the internal force coming from the particle itself is null, *i.e.* $\mathbf{G}_{ii} = 0$. The equation of motion for the i-th particle is $\mathbf{F}_i^{\text{net}} = m_i\ddot{\mathbf{r}}_i$, and by summing over i we get the equation of motion for the total system,

$$\mathbf{F} = \sum_i \mathbf{F}_i^{\text{net}} = \sum_\alpha \mathbf{F}_i + \sum_i\sum_{j\neq i}\mathbf{G}_{ij} = \sum_\alpha \mathbf{F}_i + \sum_i\sum_{j<i}(\mathbf{G}_{ij}+\mathbf{G}_{ji}) \tag{8.6}$$

$$= \sum_i m_i\ddot{\mathbf{r}}_i = \frac{d^2}{dt^2}\sum_i m_i\mathbf{r}_i = \frac{d^2}{dt^2}M\mathbf{R} = M\ddot{\mathbf{R}} = \dot{\mathbf{P}} \tag{8.7}$$

The weak form of Newton's Third Law, $\mathbf{G}_{ij} = -\mathbf{G}_{ji}$ is invoked in Eq. (8.6). What this equation of motion is saying is that:
II. *The center of mass of a system moves as if it were a single particle of mass equal to the total mass of the system, acted on by the total external force, and independent of the nature of the internal forces (as long as they follow* $\mathbf{G}_{ij} = -\mathbf{G}_{ji}$*, the weak form of Newton's Third Law).*
Combining the points we learned from **I** and **II** the following can be said:
III. *The total linear momentum for a system free of external forces is constant and equal to the linear momentum of the center of mass* (the law of conservation of linear momentum for a system).

This is precisely why your dad has to pull your sleigh on the ice. There is no way for you to both sit on the sleigh and steer it into a certain direction without you going opposite way. You and the sleigh as a whole is a system of two bodies (not particles) and whatever forces you exert on the sleigh are, however you do it, all internal forces and thus you just cannot move the sleigh in one direction with you on it, which requires the center of mass to move.

8.4 Elastic Collisions of Two Particles

For the time being, we return to two body problems and discuss elastic collisions between two particles. Although there are rarely strictly two particle collision in our real world, there are plenty of collision events that can be approximated to two particle collision, starting with billiard, a number of sports including baseball, car crashes, and etc. In all of these, there is a coordinate system that we are sitting in and observing these events. This is what we can label as a laboratory frame (LAB frame) in practice, however, for this particular exercise, we will refer to the LAB frame as the coordinate system in which the target particle is at rest.

In contrast to the LAB frame, we will also consider a zero-total-momentum coordinate system commonly known as the center-of-mass system. This is a coordinate system that moves with the center of mass of two particles so that the center-of-mass position is at rest. As a result, the net momentum of the total system is always zero, and we will refer to it as the zero-momentum frame (ZM frame).

Let's suppose a particle with mass m_1 is moving towards a particle with mass m_2 in the LAB frame. The particle 1 is moving with velocity $\mathbf{u}$ and the particle 2 is at rest. After the collision, the particle 1 and particle 2 flies off with velocities $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively. This process is shown in Fig. 8.4(a). The center-of-mass velocity is not zero in the LAB frame and we will denote it with $\mathbf{V}$.

The interaction between the two particles, either by direct contact or some force field between them, are strictly internal and therefore the center of mass momentum does not change before and after the collision. From Eq. (9.5) and conservation of momentum, we get the relation

$$\mathbf{P} = M\mathbf{V} = (m_1 + m_2)\mathbf{V} = m_1\mathbf{u} = m_1\mathbf{u}_1 + m_2\mathbf{u}_2. \tag{8.8}$$

The center-of-mass velocity is thus $\mathbf{V} = \frac{m_1}{m_1+m_2}\mathbf{u}$. What we want to know is how the particles move after the collision. In other words, we need to find out what $\mathbf{u}_1$ and $\mathbf{u}_2$ are. For this, let's turn to the ZM frame.

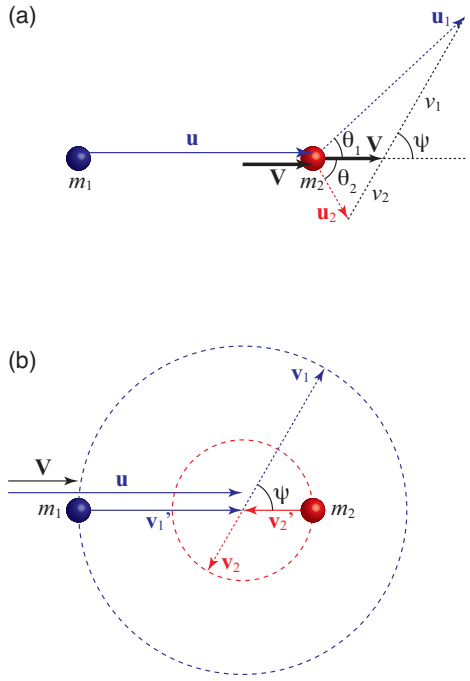

Figure 8.2:

The advantage of describing the motion in the ZM frame lies in the fact that the system as a whole, represented by the center of mass, is not moving. As stated earlier, the total momentum of the system is zero, and thus, the momentum of the two particles are always symmetric, that is, they are always of the same magnitude in the opposite direction, before or after the collision.

Since the center of mass of the two particles is moving at velocity $\mathbf{V}$ with respect to the LAB frame, the ZM frame must be moving at $\mathbf{V}$ relative to the LAB frame. In the ZM frame, the two particles are initially moving with velocities

$$\mathbf{v}_1 = \mathbf{u} - \mathbf{V} = \frac{m_2}{m_1 + m_2}\mathbf{u} \tag{8.9}$$

$$\mathbf{v}_2 = \mathbf{0} - \mathbf{V} = -\frac{m_1}{m_1 + m_2}\mathbf{u} \tag{8.10}$$

As mentioned earlier, the total momentum in the ZM frame is $\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = 0$, which is consistent with the fact that the center of mass is at rest. After the collision, the particles are moving at $\mathbf{v}_1'$ and $\mathbf{v}_2'$. From conservation of momentum, $m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1' + m_2\mathbf{v}_2'$, and conservation of kinetic energy $m_1v_1^2 + m_2v_2^2 = m_1v_1'^2 + m_2v_2'^2$, we get the relation

$$v_1 = v_1' = \frac{m_2}{m_1 + m_2}u \tag{8.11}$$

$$v_2 = v_2' = \frac{m_1}{m_1 + m_2}u = V \tag{8.12}$$

with $\mathbf{v}_1'$ and $\mathbf{v}_2'$ in the opposite directions. This is shown in Fig. 8.4(b). By the same arguments that led us to Eq. (8.9) and Eq. (8.10), now $\mathbf{u}_1$ and $\mathbf{u}_2$ can be

determined.

$$\mathbf{u}_1 = \mathbf{v}_1' + \mathbf{V} \tag{8.13}$$
$$\mathbf{u}_2 = \mathbf{v}_2' + \mathbf{V} \tag{8.14}$$

The vector diagrams Fig. 8.4 represents this relation. Finally, let's find out the relation between the deflection angles of particle 1, θ_1, particle 2, θ_2 in the LAB frame and the deflection angle of particle 1 and 2, ψ in the ZM frame. Note that the deflection angle for two particles is the same in the ZM frame and cannot exceed π. See Fig. 8.4. Between θ_1 and ψ the following relation holds:

$$\tan\theta_1 = \frac{v_1'\sin\psi}{V + v_1'\cos\psi} = \frac{\sin\psi}{(V/v_1') + \cos\psi} = \frac{\sin\psi}{(m_1/m_2) + \cos\psi}. \tag{8.15}$$

Here Eq. (8.11) and Eq. (8.12) are used to obtains the relation $V/v_1' = m_1/m_2$. Similarly for θ_2,

$$\tan\theta_2 = \frac{v_2'\sin\psi}{V - v_2'\cos\psi} = \frac{\sin\psi}{(V/v_2') - \cos\psi} = \frac{\sin\psi}{1 - \cos\psi} = \cot(\psi/2). \tag{8.16}$$

Otherwise, from the geometry, one can easily show that

$$\theta_2 = \frac{1}{2}(\pi - \psi) \tag{8.17}$$

which is an identical result.

We can define the lab opening angle, θ, as $\theta_1 + \theta_2$ and

$$\tan\theta = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} = \left(\frac{m_1 + m_2}{m_1 - m_2}\right)\cot(\psi/2) \tag{8.18}$$

8.5 Kinematics of Elastic Collisions

Let's focus on the energy relationship between the particles. In the LAB frame, the total energy of the particles, E_0 is conserved and is equal to

$$E_0 = \frac{1}{2}m_1u^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 = E_1 + E_2. \tag{8.19}$$

The ratio between E_2 and E_0 is

$$\frac{E_2}{E_0} = \frac{\frac{1}{2}m_2u_2^2}{\frac{1}{2}m_1u^2} = \frac{m_2u_2^2}{m_1u^2}. \tag{8.20}$$

From the relation, (see Fig. 8.4)

$$v_2' = 2V\sin(\psi/2) \tag{8.21}$$

we get

$$\frac{E_2}{E_0} = \frac{4m_1m_2}{(m_1 + m_2)^2}\sin^2(\psi/2) \tag{8.22}$$

8.6 Inelastic Collisions

In the real world, not all collisions are elastic. In such a case, total kinetic energy of the particles is not conserved, and energy is conserved through

$$Q + \frac{1}{2}m_1u^2 = \frac{1}{2}m_1u_1^2 + \frac{1}{2}m_2u_2^2 \tag{8.23}$$

where Q is called the Q-value and represents the energy loss or gain in the collision.

$Q = 0$: Elastic collision, kinetic energy is conserved.

$Q > 0$: Exoergic collision, kinetic energy is gained.

$Q < 0$: Endoergic collision, kinetic energy is lost.

An inelastic collision is an endoergic process, through which a part of kinetic energy is converted to mass, heat, sound energy and etc.

There is a fairly simple relation between particle velocities in the direction normal to the plane of contact that allows us to determine if a collision is elastic or not. When the ratio known as the coefficient of restitution

$$\epsilon = \frac{|u_2 - u_1|}{u} \tag{8.24}$$

is equal to one, the collision is perfectly elastic. When $\epsilon = 0$, the collision is totally inelastic.

Any collision whether elastic or inelastic causes momentum of each particle (but not of the two-particle system as a whole) to change. This change in momentum is due to the impulsive forces $\mathbf{F} = \frac{d}{dt}(m\mathbf{v})$ acting on the particles throughout the time period $\Delta t = t_2 - t_1$ of the collision. We define impulse $\mathbf{P}$ as $\int_{t_1}^{t_2} \mathbf{F}dt \equiv \mathbf{P}$.

8.7 Reduced Mass

So far, we have treated multi-particle systems including two-body collision as multi-particle systems. However, for the case of two-body system, one can reduce the problem in to a one body problem that we are familiar with.

Imagine two objects with masses m_1 and m_2 positioned at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively. These two objects are imposing a force to each other along the line connecting them. Let's assume that the force can be described by a potential $V(r)$ depending only on the distance between the two objects $r = |\mathbf{r}_1 - \mathbf{r}_2|$. The Lagrangian for the system is

$$\mathcal{L} = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - V(r) \tag{8.25}$$

As a general rule, in describing the dynamics of two bodies in three dimensional space, we need to identify six independent coordinates (three for each body) and find time dependence of these six coordinates, *e.g.* r_{1x}, r_{1y}, r_{1z}, r_{2x}, r_{2y} and r_{2z}. Setting up Lagrangian equations based on these coordinates yields

$$m_1\ddot{\mathbf{r}}_1 = -\nabla_1 V(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{8.26}$$

$$m_2\ddot{\mathbf{r}}_2 = -\nabla_2 V(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{8.27}$$

where ∇_i represents the gradient operator with respect to the $\mathbf{r}_i$ coordinate frame. These are simply Newtonian equations of motion, and as familiar as they are in looks, solving the equations in general is not easy.

Of course, a better choice of generalized coordinates can turn this into a simpler problem. One such coordinate transformation is

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{8.28}$$

where $\mathbf{R}$ is the center of mass position and $\mathbf{r}$ the relative position between the two objects. The Lagrangian in terms of $\mathbf{R}$ and $\mathbf{r}$ is

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}|\dot{\mathbf{r}}|^2 - V(r) \tag{8.29}$$

The above Lagrangian is completely separable with respect to the two variables $\mathbf{R}$ and $\mathbf{r}$, that is

$$\mathcal{L}(\mathbf{r}, \mathbf{R}, \dot{\mathbf{r}}, \dot{\mathbf{R}}) = \mathcal{L}_{\text{CM}}(\mathbf{R}, \dot{\mathbf{R}}) + \mathcal{L}_{\text{rel}}(\mathbf{r}, \dot{\mathbf{r}}) \tag{8.30}$$

$$\mathcal{L}_{\text{CM}}(\mathbf{R}, \dot{\mathbf{R}}) = \frac{1}{2}M\dot{\mathbf{R}}^2 \tag{8.31}$$

$$\mathcal{L}_{\text{rel}}(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - V(r) \tag{8.32}$$

where $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$. We can then solve for the motion of $\mathbf{R}$ and $\mathbf{r}$ as two separate problems.

Solving for Euler-Lagrange equation from $\mathcal{L}_{\text{CM}}$ just results in

$$M\ddot{\mathbf{R}} = 0 \qquad \text{or} \qquad \dot{\mathbf{R}} = \text{constant} \tag{8.33}$$

This means that the center of mass moves at a constant velocity. This is consistent with the fact that there is no net force acting on the two body as a whole system. As each particles are under the influence of the central force, momentum of the individual particle is not conserved, but the central force field provides only internal force and the momentum of the whole system is conserved. A frame moving at a constant velocity is an inertial frame, thus the physics remains unchanged in a frame moving with the center of mass. In such a frame $\mathbf{R} = 0$ and we get $\mathcal{L} = \mathcal{L}_{rel}$.

If we now turn our attention to the component with the relative coordinate $\mathbf{r}$, we can see that the Lagrangian

$$\mathcal{L}_{rel} = \frac{1}{2}\mu|\dot{\mathbf{r}}|^2 - V(r) \tag{8.34}$$

is the same form as the one given for a single object as in Eq. (7.5). In other words, an appropriate choice of coordinate system can reduce the two body m_1-m_2 problem into a one body problem with a reduced mass of $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

One should not be confused about this reduction as a two body conglomerated into a single body with mass μ, or as one body fixed and the other body moving around the fixed body with an effective mass of μ. There is no physical body moving around with mass μ in this problem. What the reduction is saying is that if we are willing to just focus on the relative position $\mathbf{r}$, we can imagine a particle with mass μ moving in a central force field $V(\mathbf{r})$ and solving for this

problem will eventually allow us to understand the original two-body problem in the following way.

Once we identify the time dependent motion of $\mathbf{r}(t)$, we can describe the motion of the actual two physical bodies according to the coordinate transformation,

$$\mathbf{r}_1 = \frac{m_2}{m_1+m_2}\mathbf{r} + \mathbf{R}, \quad \mathbf{r}_2 = -\frac{m_1}{m_1+m_2}\mathbf{r} + \mathbf{R} \tag{8.35}$$

following from Eq. (8.28).

8.8 Scattering Cross Section

So far we have discussed kinematics of scattering, *i.e.* relationship between the initial state before a collision and the final state after it. The details of the interaction between two colliding particles were not taken into account. We now focus on that missing detail which specifies the kinematic relation. Despite our tendency to think of a collision as two objects coming in direct contact and bouncing off, such direct contact is not required for a *collision* or *scattering* to take place. A force field between two particles can cause the particles to deflect from their original trajectory and, in general, any such interactions can be called scattering. In fact, even the collision through direct contact can be understood with the force field that only acts at a single distance of hard-core repulsion. Here we impose one assumption that any force field between these colliding particles are vanishingly small at large distances, $V(r) \to 0$ as $r \to \infty$.

Imagine, as shown in Fig. 8.8, the incident particle, m_1, moving towards the target particle, m_2. Without any interacting force field between the particles, m_1 may get as close as the distance p away from m_2 but may not actually come to contact with m_2. If a repulsive interaction, such as Coulomb interaction between the same-charge particles, is present, m_1 will be deflected away from the original path and the minimum distance between the two particles will be greater than p. Nonetheless, this imaginary minimum distance, p, between the two particles is what we call the **impact parameter**. The impact parameter has a direct relation to the deflection angles θ_1 and θ_2 and our task is to find that relationship.

In this section we will study the collisions due to central force field, and it is helpful to first reduce the two body problem into a one body problem. This can be done by following the procedure from the previous section. Instead of a particle with mass m_1 flying in with initial velocity $\mathbf{u}$ and impact parameter p towards the stationary particle with mass m_2, we will consider the motion of a particle with mass μ with initial velocity $\mathbf{u}$ and impact parameter p towards the origin of a central force field $V(r)$. One can easily show why the initial velocity and the impact paramter of μ are unaffected by the reduction. We will leave it as an exercise for readers. The new problem can be viewed as Fig. 8.8. For a well-defined force field, there should be one-to-one correspondence between the impact parameter p and the scattering angle ϕ. In other words, an incident particle flying in with a certain impact parameter p will always scatter off at an angle ϕ.

To find this relation, let's first identify the angular momentum and total

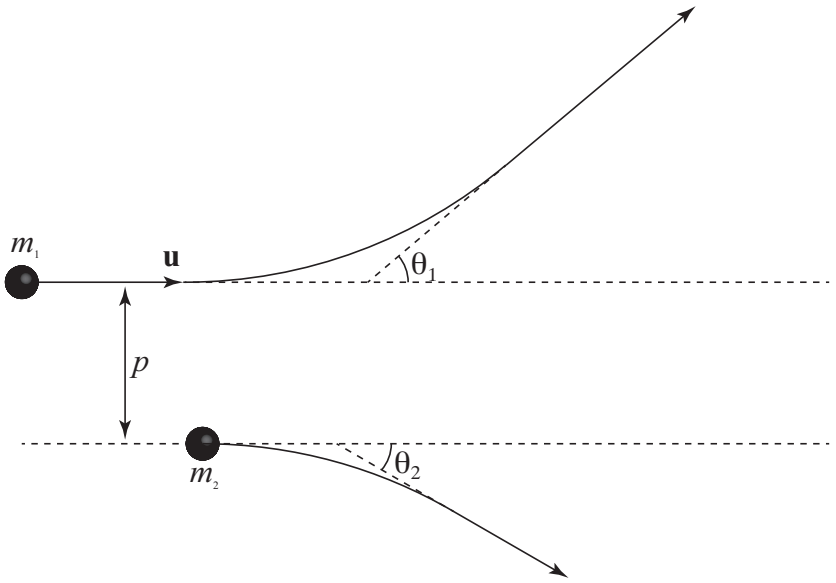

Figure 8.3:

energy of the particle in the reduced mass frame,

$$l = \mu u p \tag{8.36}$$

$$E = \frac{1}{2}\mu u^2 \tag{8.37}$$

The scattering angle can be determined by finding the orbit or path of the particle. For a particle moving under the influence of central force given by the potential $V(r)$, we can find the angle Θ between the maximum distance position and the minimum distance position (Fig. 8.8) from the relation

$$\Theta = \int_{r_{min}}^{r_{max}} \frac{(l/r^2)dr}{\sqrt{2\mu[E - V(r) - (l^2/2\mu r^2)]}}. \tag{8.38}$$

In this case, the orbit is a particle at an infinite distance, *i.e.* $r_{max} = \infty$, approaching the center of the force field and then moving away to an infinite distance as shown in Fig. 8.8. Simple geometric calculation leads to $\phi = \pi - 2\Theta$. Combining equations (8.36), (8.37) and (8.38) , we get

$$\psi = \pi - 2\int_{r_{min}}^{\infty} \frac{(p/r^2)dr}{\sqrt{1 - (p^2/r^2) - (2V/\mu u^2)}}. \tag{8.39}$$

which becomes a function of p and u if the central force potential $V(r)$ is given. In other words, there is a definite relation between the deflection angle ϕ, and the initial velocity u and impact parameter p. When we are dealing with macroscopic particles such as billiard balls, all these can be extracted with a reasonable accuracy and the interaction potential $V(r)$ can be extracted. However, what we can observe from the LAB frame is not the angle ϕ, but θ_1 and θ_2. Therefore, we need to be able to translate θ_is into ϕ in order to make this comparison.

Rather fortunately, it turns out that the angle ϕ is the same as the scattering angle, ψ, in the ZM frame discussed in section 8.4. Once again, one should not

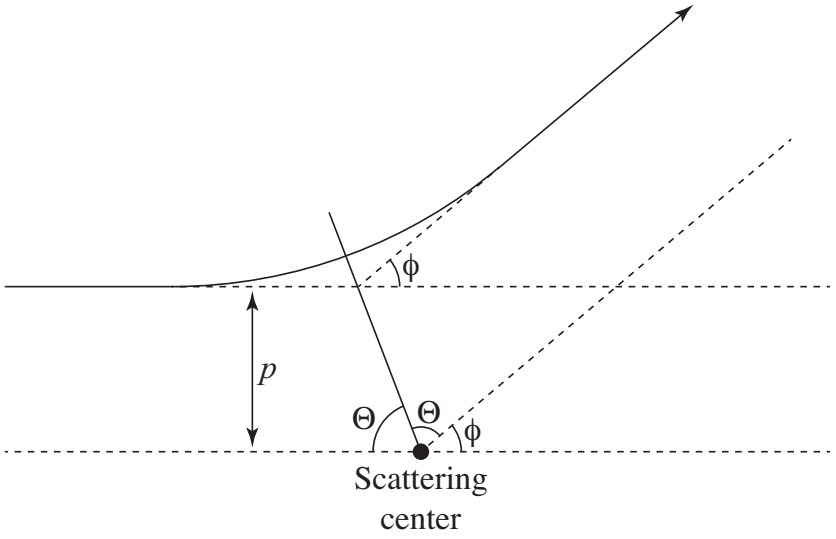

Figure 8.4:

be confused by the coincidence that the reduced mass frame is identical to the ZM frame. This is merely a consequence of the fact that the direction of $\mathbf{r}$ is not changed from the reduced mass frame to the ZM frame and hence the angle ϕ is preserved. So, from now on, we can treat the angle ϕ as ψ and translation between θ_is and ϕ can be obtained by Eq. (8.15) and Eq. (8.17).

When we are dealing with subatomic particles, *e.g.* in the case of Rutherford scattering, the energy of the particles can be known independently. However, it is extremely hard to fire a single particle with an exact impact parameter p. In many cases, we end up shooting a beam of particles. Even if we try our best, the beam will not be a single line, but will be a bundle that has some cross sectional area. In such a case, it is useful to define **flux density**, N, as the number of incident particles per unit area per unit time.

Once these incident particles come under the influence of the force field, they will scatter in different directions. Since a particle with a given impact parameter p always flies out with a scattering angle ψ, if we consider a small variation dp in the impact parameter, the scattered angle will vary by a small amount $d\psi$. If there is a beam of incident particles that passes through a small ring of width dp about the impact parameter p, all these particles will scatter off into a solid angle formed by rotating $d\psi$ about the axis of azimuthal symmetry as shown in Fig. 8.8, $d\Omega$. Since the force field is assumed to be central, the scattering process is symmetric about the azimuthal axis, and therefore the solid angle element is simply $d\Omega = 2\pi \sin\psi \, d\psi$. In other words, there is a post-scattering cross sectional area that corresponds to the pre-scattering cross section $2\pi p dp$. To calculate the post-scattering cross section, we define a **differential scattering cross section**, $\sigma(\psi)$, which should hold the following relation

$$|\sigma(\psi) d\Omega| = |2\pi \, p \, dp| \tag{8.40}$$

From this, we obtain the relation between the scattering cross section and

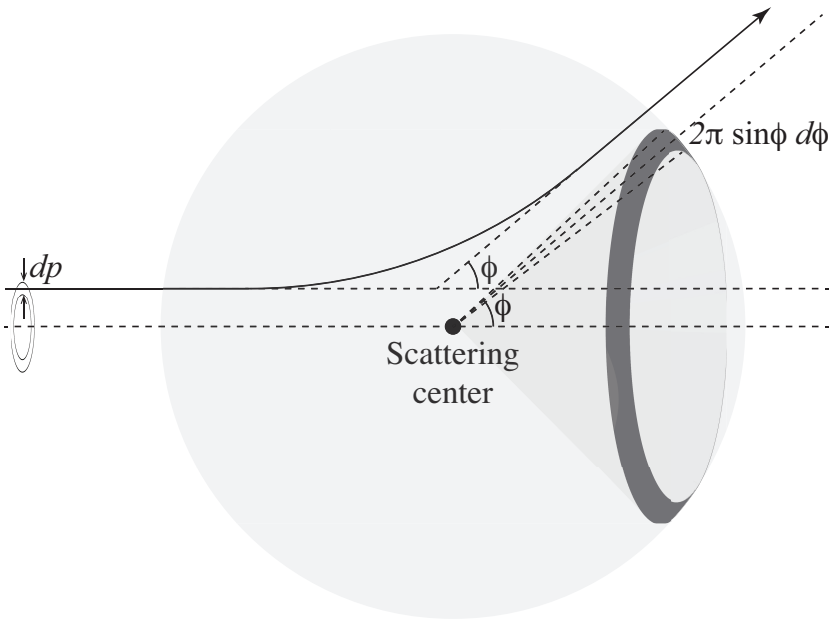

Figure 8.5:

the impact parameter

$$\sigma(\psi) = \frac{p}{\sin\psi}\left|\frac{dp}{d\psi}\right|. \tag{8.41}$$

However, we need to know $|dp/d\theta|$ to actually calculate the scattering cross section, and for this, we turn to Eq. (8.39). By inverting $\psi = \psi(p)$, we can calculate $p = p(\psi)$ and the differential scattering cross section $\sigma(\psi)$ can be expressed purely as a function of ψ with Eq. (8.41).

However, the cross section we have calculated is in the reduced mass frame, and most of the observations are made in the LAB frame. We need to make a transformation from the reduced mass frame to the LAB frame. Because ψ is identical to the scattering angle in the ZM frame, we can follow the analysis from the Section 8.4 for this. The total number of particles scattered into a unit solid angle must be the same in the LAB frame as in the ZM frame, and we have

$$\sigma(\psi)d\Omega = \bar{\sigma}(\theta_1)d\Omega' \tag{8.42}$$

$$\sigma(\psi)2\pi\sin\psi\, d\psi = \bar{\sigma}(\theta_1)2\pi\sin\theta_1\, d\theta_1 \tag{8.43}$$

Here θ_1 and ψ represent the same scattering angle in two different coordinate system, *i.e.* the LAB frame and the ZM frame, respectively. Similarly $d\Omega'$ and $d\Omega$ represent the same element of solid angle in the two coordinate systems. Thus $\bar{\sigma}(\theta_1)$ is the differential cross sections for the scattering in the LAB frame.

$$\bar{\sigma}(\theta_1) = \sigma(\psi)\frac{\sin\psi}{\sin\theta_1}\frac{d\psi}{d\theta_1} = \sigma(\psi)\frac{d\cos\psi}{d\cos\theta_1} \tag{8.44}$$

The bar over σ indicates that the angle dependence or the functional form of

$\bar{\sigma}(\theta_1)$ is not identical to that of $\sigma(\psi)$. Eventually, one should be able to derive

$$\frac{d\cos\psi}{d\cos\theta_1} = \frac{d}{d\cos\theta_1}\left[\frac{m_1}{m_2}\left(-\sin^2\theta_1 \pm \cos\theta_1\sqrt{\left(\frac{m_2}{m_1}\right)^2 - \sin^2\theta_1}\right)\right] \quad (8.45)$$

$$= \pm\frac{[\gamma\cos\theta_1 \pm \sqrt{1-\gamma^2\sin\theta_1}]^2}{\sqrt{1-\gamma^2\sin\theta_1}} \quad (8.46)$$

where $\gamma \equiv m_1/m_2$. Also by inverting the Eq. (8.15), we can derive $\psi = \sin^{-1}(\gamma\sin\theta_1) + \theta_1$. Substituting ψ for this relation in $\sigma(\psi)$ for Eq. (8.44), we get $\bar{\sigma}(\theta_1)$ purely as a function of θ_1.

8.9 Rutherford Scattering Formula

We take a look at a very special case of the particle scattering we have discussed so far. Hans Geiger and Enest Marsden in 1909 struck a gold foil with a beam of α-particles, *i.e.* He^{2+}, under the direction of Ernest Rutherford. Although it was not fully understood at the moment, the interaction between the α-particles and the gold foil is mostly due to Coulomb interaction between the α-particles and the positively charged nuclei of gold obeying the relation $V = \frac{k}{r} \equiv \frac{q_1 q_2}{4\pi\epsilon_0 r}$ with q_1 and q_2 the amounts of charge that the two interacting particles carry.

Substituting $V = k/r$ in Eq. 9.63, we get

$$\psi = \pi - 2\int_{r_{min}}^{\infty} \frac{(b/r)dr}{\sqrt{r^2 - (k/E)r - b^2}}. \quad (8.47)$$

Carrying out the integration in the right hand side with $\kappa \equiv \frac{k}{2E}$ results in the relation

$$\cos\Theta = \frac{\kappa/b}{\sqrt{1+(\kappa/b)^2}} \quad (8.48)$$

Rewriting Eq. 9.68 as $\tan\Theta = b/\kappa$ and using the relation $\psi = \pi - 2\Theta$, b is expressed in terms of ψ: $b = k\cot(\psi/2)$. The scattering cross section is

$$\sigma(\psi) = \frac{\kappa^2}{2}\frac{\cot(\psi/2)}{\sin\psi\sin^2(\psi/2)} = \frac{\kappa^2}{2}\frac{1}{\sin^4(\psi/2)} = \frac{k^2}{(4E)^2}\frac{1}{\sin^4(\psi/2)} \quad (8.49)$$

8.10 Angular Momentum

We now turn our focus to the angular momentum of multi-particle systems. Unlike the linear momentum, the angular momentum has to be defined with respect to a point. The total angular momentum about the origin of a given coordinate system O, $\mathbf{L}_O$, is the sum of angular momentum of all the individual particles, $\mathbf{l}_i = \mathbf{r}_i \times \mathbf{p}_i = \mathbf{r}_i \times (m_i\dot{\mathbf{r}}_i)$, in the system.

$$\mathbf{L}_O = \sum_i \mathbf{L}_i = \sum_i \mathbf{r}_i \times (m_i\dot{\mathbf{r}}_i) \quad (8.50)$$

Earlier, we have shown that the rate of change in total angular momentum of the system about a point O is equal to the net resultant external torques about that point, that is,

$$\frac{d\mathbf{L}_O}{dt} = \mathbf{K}_O \quad (8.51)$$

We can also define angular momentum of the system with respect to a general point A rather than O. The position vector of the i-th particle about A is given by $\mathbf{r}_i - \mathbf{a}$ and assuming that the point A is fixed within the coordinate frame, $\mathbf{L}_A$ becomes

$$\begin{aligned} \mathbf{L}_A &= \sum_i (\mathbf{r}_i - \mathbf{a}) \times (m_i \dot{\mathbf{r}}_i) = \sum_i \{\mathbf{r}_i \times (m_i \dot{\mathbf{r}}_i)\} - \mathbf{a} \times \sum_i m_i \dot{\mathbf{r}}_i \\ &= \mathbf{L}_O - \mathbf{a} \times \mathbf{P} \end{aligned} \tag{8.52}$$

with the total linear momentum of the system, $\mathbf{P}$, is identified as Eq. (8.5). The angular momentum, $\mathbf{L}_A$, of a particle, P, with respect to a point, A, is always perpendicular to the plane containing A, P and the velocity of the particle $\dot{\mathbf{r}}$. The component of the angular momentum about an arbitrary axis $\mathbf{n}$ by projecting the angular momentum vector $\mathbf{L}_A$ onto $\mathbf{n}$:

$$\mathbf{L}_A \cdot \mathbf{n} = \rho \mathbf{p} \cdot \hat{\boldsymbol{\phi}} \tag{8.53}$$

where ρ is the perpendicular distance of the particle from the axis and $\mathbf{p} \cdot \hat{\boldsymbol{\phi}}$ is the azimuthal component of $\mathbf{p}$ around the axis.

If we let the point A move around inside our coordinate frame, then

$$\begin{aligned} \mathbf{L}_A &= \sum_i (\mathbf{r}_i - \mathbf{a}) \times m_i (\dot{\mathbf{r}}_i - \dot{\mathbf{a}}) \\ &= \sum_i \{\mathbf{r}_i \times (m_i \dot{\mathbf{r}}_i)\} - \mathbf{a} \times \sum_i m_i \dot{\mathbf{r}}_i - \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{a}} + \sum_i m_i \mathbf{a} \times \dot{\mathbf{a}} \\ &= \mathbf{L}_O - \mathbf{a} \times \mathbf{P} - M\mathbf{R} \times \dot{\mathbf{a}} + M\mathbf{a} \times \dot{\mathbf{a}} \end{aligned} \tag{8.54}$$

From this, we can invoke an interesting feature about angular momentum. If we choose our point A to be the center of mass of the system, G, the last two terms cancel each other, and we are left with the relation

$$\mathbf{L}_O = \mathbf{L}_G + \mathbf{R} \times \mathbf{P} \tag{8.55}$$

Total angular momentum about an origin is the sum of the angular momentum of the center of mass about that origin and the angular momentum of the system about the position of the center of mass. From this relation, we can also deduce that

$$\frac{d\mathbf{L}_G}{dt} = \mathbf{K}_G \tag{8.56}$$

In any motion of a system $\mathcal{S}$, the rate of increase of the angular momentum about the CM is equal to the total moment about the CM of the external forces acting on $\mathcal{S}$, regardless of the motion of the center of mass.

8.11 Energy of the System

In chapter 2, we've learned that the total energy, $E = T + V$, of a particle in a conservative force field is a constant, where T is the kinetic energy and U is the potential energy. Is this true for a system of many particles? If the system gets rearranged from a certain configuration 1 to a different configuration 2 by

the total force acting on the system, that means some work, W_{12}, is done on the system,

$$W_{12} = \sum_i \int_1^2 \mathbf{F}_i^{\text{net}} \cdot d\mathbf{r}_i = \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i + \sum_i \sum_{j\neq i} \int_1^2 \mathbf{G}_{ij} \cdot d\mathbf{r}_i \quad (8.57)$$

$$= \sum_i \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i + \sum_{i<j} \left(\int_1^2 \mathbf{G}_{ij} \cdot d\mathbf{r}_i + \int_1^2 \mathbf{G}_{ji} \cdot d\mathbf{r}_i \right) \quad (8.58)$$

$$= -\sum_i \int_1^2 \nabla_i V_i \cdot d\mathbf{r}_i - \sum_{i<j} \left(\int_1^2 \nabla_i \bar{V}_{ij} \cdot d\mathbf{r}_i + \int_1^2 \nabla_j \bar{V}_{ji} \cdot d\mathbf{r}_i \right) \quad (8.59)$$

$$= -\sum_i \int_1^2 dV_i - \sum_{i<j} \int_1^2 d\bar{V}_{ij} = \left[-(\sum_i V_i + \sum_{i<j} \bar{V}_{ij}) \right]_1^2 \quad (8.60)$$

Going from Eq. (8.58) to Eq. (8.57), we used the assumption that the force field is conservative, that is, the forces $\mathbf{F}_i$ and $\mathbf{G}_{ij}$ are derivable from potential functions V_i and $\bar{V}_{ij}$, respectively. We can define the total potential energy $V = -(\sum_i V_i + \sum_{i<j} U_{ij})$, and we get

$$W_{12} = V_1 - V_2. \quad (8.61)$$

In terms of kinetic energy, the work done on the system is

$$W_{12} = \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right) = T_2 - T_1. \quad (8.62)$$

where $T = \sum_i \frac{1}{2} m_i v_i^2$. Combining Eq. (8.57) and Eq. (8.62), we get $W_{12} = v_1 - V_2 = T_2 - T_1$ and thus $E = V_1 + T_1 = V_2 + T_2$. This means: *The total energy for a conservative system is constant.*

The kinetic energy, when written out, is

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i |\dot{\mathbf{r}}_i|^2 = \sum_i \frac{1}{2} m_i (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i') \cdot (\dot{\mathbf{R}} + \dot{\mathbf{r}}_i') \quad (8.63)$$

$$= \sum_i \frac{1}{2} m_i (\dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \dot{\mathbf{r}}_i' \cdot \dot{\mathbf{r}}_i' + 2\dot{\mathbf{R}} \cdot \dot{\mathbf{r}}_i') \quad (8.64)$$

$$= \sum_i \frac{1}{2} m_i V^2 + \sum_i \frac{1}{2} m_i v_i'^2 + \dot{\mathbf{R}} \cdot \frac{d}{dt} (\sum_i m_i \mathbf{r}_i') \quad (8.65)$$

$$= \frac{1}{2} M V^2 + \sum_i \frac{1}{2} m_i v_i'^2 \quad (8.66)$$

which can be stated: *The total kinetic energy of the system is equal to the sum of the kinetic energy of a particle of mass M moving with the velocity of the center of mass and the kinetic energy of motion of the individual particles relative to the center of mass.*

Finally we return to the problem of high jumping. A human body, of course, is a system of particles. When we high jump, we kick off the ground and gain some momentum in vertical direction. The jumper will have kinetic energy associated with that momentum and it will allow the jumper's center of mass

to reach a certain height. Ignoring small amount of kinetic energy associated with moving body parts here and there, the final height of the jumper's center of mass is more or less given by the amount of momentum he or she picks up at the time of kicking off the ground. So the most effective way to high jump would be to find a way to put your center of mass outside your body and have your body parts actually role around the center of mass. That is exactly what these high jumpers are doing.

The most traditional method known as the scissors technique looks quite natural and easy, but the center of mass actually has to clear the height of the bar and thus is not very effective. Straddle technique or Fosbury flop are more advanced techniques and lets you clear higher bars because the center of mass actually goes under the bar, and therefore widely used these days.

Chapter 9

Rigid Body Kinematics

In this chapter, we will turn our attention away from particles and focus on rigid bodies. The rigid body is defined to be a system in which the distance between any pair of composing particles is fixed, that is, for any two points, P_i and P_j, on the rigid body represented by position vectors $\mathbf{r}_i$ and $\mathbf{r}_j$, the following relationship holds for all time t

$$|\mathbf{r}_i - \mathbf{r}_j| = c_{ij} \tag{9.1}$$

where the c_{ij} are constant. One can show that the above definition of rigidity is equivalent to

$$\dot{\mathbf{r}}_i \cdot \mathbf{r}_j + \mathbf{r}_i \cdot \dot{\mathbf{r}}_j = 0 \tag{9.2}$$

For a rigid body, the main feature in understanding its dynamics heavily relies on distinguishing translational and rotational motions.

9.1 Rotation and Linear Velocity

In two dimensional space, rotation axis is always perpendicular to the plane of motion, and angular velocity can be treated as a scalar. However, in three dimensional space, the rotation axis can take arbitrary direction, say $\hat{\mathbf{n}}$ and it is useful to identify (right-handed) angular velocity as a vector along the rotation axis,

$$\boldsymbol{\omega} = \omega\hat{\mathbf{n}} \tag{9.3}$$

We will leave proving that the angular velocity can be represented by a vector as an exercise for the readers. The fact that the angular velocity is a vector has a rather interesting implications, which is that the angular velocity can be decomposed into two or more vector components:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2. \tag{9.4}$$

At first glance, this might seem like a rather unintuitive and unnecessary property of the angular velocity, but it will become extremely useful later.

With the angular velocity represented as a vector, the linear velocity of any point, P, within a rotating rigid body can be obtained by

$$\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{b}) \tag{9.5}$$

where $\mathbf{r}$ is the position vector of P and $\mathbf{b}$ is the position vector of any fixed point, B, on the rotation axis. In other words, the point P has a relative velocity with respect to the point B given by $\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{b})$. Here, one should note that the rotation axis including the point B does not necessarily have to go through the rigid body.

If the point B has a translational velocity $\mathbf{v}^B$ with respect to some fixed coordinate system, because of the definition of rigidity, every point on the body must have the same translational velocity $\mathbf{v}^B$. Then the net linear velocity of the point P should be the vector sum of $\mathbf{v}^B$ and $\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{b})$:

$$\mathbf{v} = \mathbf{v}^B + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{b}) \tag{9.6}$$

The above relation results in a rather interesting property of the rigid body. Let us consider two different points A and P, then the velocity of these two points can be given by

$$\mathbf{v}^A = \mathbf{v}^B + \boldsymbol{\omega} \times (\mathbf{a} - \mathbf{b}) \tag{9.7}$$
$$\mathbf{v} = \mathbf{v}^B + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{b}) \tag{9.8}$$

where $\mathbf{a}$ is the position vector of A. Subtracting Eq. 9.7 from Eq. 9.8, we get

$$\mathbf{v} = \mathbf{v}^A + \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a}) \tag{9.9}$$

What this states is that the velocity of the point P on the rigid body can be decomposed to the translational velocity of any point A, $\mathbf{v}^A$ within the rigid body and the rotational velocity of P with respect to A, $\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a})$. One should pay attention to the fact that the point A can be any point within the rigid body not necessarily along the rotating axis originally considered. In this regard, it is fair to say that there is no single fixed rotating axis in a rigid body. Any point on the rigid body is rotating about another point on rigid body with the identical angular velocity $\boldsymbol{\omega}$.

9.2 Moment of Inertia Tensor

As we have seen in the previous section, velocity of particles within a rigid body can be dissected into the linear velocity of a reference point and rotational velocity with respect to that reference point. This feature allows us to interpret a motion of a rigid body in a dramatically simpler manner.

The ultimate goal in mechanics is to be able to find the positions and velocities of particles as functions of time. The fact that the velocity of a rigid body with respect to a reference point is simply rotational velocity given by $\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a})$ in Eq. (9.9) indicates that when a motion of any single particle within the rigid body is given, all we are left is to find the rotational motion about that point.

Thus, the starting point in rigid body dynamics naturally becomes writing out the angular momentum:

$$\mathbf{L}_O = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha \tag{9.10}$$

where $\mathbf{r}_\alpha$ is the position of α-th particle relative to the fixed point. If the angular momentum is defined with respect to the reference point of the rotational

motion, the linear momentum of the α-th particle, $\mathbf{p}_\alpha$, can be expressed as $\mathbf{p}_\alpha = m_\alpha \boldsymbol{\omega} \times \mathbf{r}_\alpha$ and therefore

$$\mathbf{L}_O = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha). \tag{9.11}$$

Using the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{A}) = A^2\mathbf{B} - \mathbf{A}(\mathbf{A} \cdot \mathbf{B})$ (or $[\mathbf{A} \times (\mathbf{B} \times \mathbf{A})]_i = B_i A^2 - A_i(\mathbf{A} \cdot \mathbf{B})$), the i-th component of $\mathbf{L}_O$ is expressed as

$$\begin{aligned} L_i &= \sum_\alpha m_\alpha \left(\omega_i \sum_k r_{\alpha,k}^2 - r_{\alpha,i} \sum_j r_{\alpha,j}\omega_j \right) \\ &= \sum_\alpha m_\alpha \sum_j \left(\omega_j \delta_{ij} \sum_k r_{\alpha,k}^2 - \omega_j r_{\alpha,i} r_{\alpha,j} \right) \\ &= \sum_j \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k r_{\alpha,k}^2 - r_{\alpha,i} x_{\alpha,j} \right) \end{aligned} \tag{9.12}$$

where the relation $\omega_i = \sum_j \delta_{ij}\omega_j$ is used going from the second line to the third line. The δ_{ij} is defined as

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \tag{9.13}$$

The Greek lettered indices α, β, γ, etc. represent each particle and the Italic indices i, j and k represent the three components in Cartesian coordinate system, that is, $r_{\alpha,i}$ is the i-th Cartesian component of the α-th particle, or $r_{\alpha,1} = x_\alpha$, $r_{\alpha,2} = y_\alpha$ and $r_{\alpha,3} = z_\alpha$.

If we define I_{ij} as

$$I_{ij} \equiv \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k r_{\alpha,k}^2 - r_{\alpha,i} x_{\alpha,j} \right), \tag{9.14}$$

we can rewrite the Eq. (9.12):

$$L_i = \sum_j I_{ij}\omega_j \tag{9.15}$$

The above equation can be rewritten in matrix notation, $\mathbf{L} = \mathbb{I} \cdot \boldsymbol{\omega}$. Here, $\mathbf{L}$ and $\boldsymbol{\omega}$ are three dimensional vectors representing the angular momentum and angular velocity, respectively, and $\mathbb{I}$ is a three-by-three matrix. The elements of the matrix $\mathbb{I}$ denoted as I_{ij} are referred to as the moment of inertia tensors. They represent the moment of inertia associated with the angular velocity in the j-th component that contributes to angular momentum in the i-th component.

From the Eq. (9.12), we can see that the angular momentum along the r_i-axis is a result of rotation along all three x, y and z-axes, ω_x, ω_y and ω_z. In other words, a general rotation given by an angular velocity in three dimensional space is not necessarily colinear with the angular momentum. In describing a rotational motion, the moment of inertia must be able to identify such non-colinearity, which is done by utilizing tensor representation.

9.3 Kinetic Energy

The kinetic energy of the rigid body can now be calculated from

$$T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2 = \frac{1}{2} m_\alpha (\boldsymbol{\omega} \times \mathbf{r}_\alpha)^2 \tag{9.16}$$

and the total kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} \sum_\alpha m_\alpha (\boldsymbol{\omega} \times \mathbf{r}_\alpha)^2 \qquad (9.17) \\ &= \frac{1}{2} \sum_\alpha m_\alpha \left[\left(\sum_i \omega_i^2 \right) \left(\sum_\gamma r_{\alpha,k}^2 \right) - \left(\sum_i \omega_i r_{\alpha,i} \right) \left(\sum_j \omega_j r_{\alpha,j} \right) \right] \end{aligned}$$

where the relation $(\mathbf{A} \times \mathbf{B})^2 = A^2 B^2 - (\mathbf{A} \cdot \mathbf{B})^2$ was used to get the second line from the first line.

Now, we can write $\omega_i = \sum_j \omega_j \delta_{ij}$, so that

$$\begin{aligned} T &= \frac{1}{2} \sum_\alpha \sum_{ij} m_\alpha \left[\omega_i \omega_j \delta_{ij} \left(\sum_k r_{\alpha,k}^2 \right) - \omega_i \omega_j r_{\alpha,i} r_{\alpha,j} \right] \\ &= \frac{1}{2} \sum_{ij} \omega_i \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k r_{\alpha,k}^2 - r_{\alpha,i} r_{\alpha,j} \right) \\ &= \frac{1}{2} \sum_{ij} I_{ij} \omega_i \omega_j \qquad (9.18) \end{aligned}$$

This is the general form for the rotational kinetic energy for a three dimensional rigid body. Using the matrix form, this can also be expressed as,

$$T = \frac{1}{2} \boldsymbol{\omega}^T \cdot \mathbb{I} \cdot \boldsymbol{\omega} = \frac{1}{2} \boldsymbol{\omega}^T \cdot \mathbf{L}. \tag{9.19}$$

where the row vector $\boldsymbol{\omega}^T$ is the transpose of a column vector $\boldsymbol{\omega}$.

9.4 Rigid Body Kinematics

With the general form of the angular momentum given by Eq. (9.15), in principle we can solve for the (torque) equation of motion $\mathbf{K}_O = d\mathbf{L}_O/dt$. However, it is not immediately obvious why defining the moment of inertia tensors are helpful at all. The moment of inertia tensors contain coordinates of particles, $\mathbf{r}_\alpha$. In other words, if the particles move around in space, which they do in the system of our interest, the moment of inertia tensors continuously change with the motion of the particles. They can be fully determined only after the equation of motion is solved.

This dilemma is resolved once again by the definition of rigid body. If we think of a co-rotating coordinate system with the rigid body, usually referred to as the body coordinate, the positions of each particle do not change in this coordinate. In other words, the moment of inertia tensors in this co-rotating coordinate are time independent. Also, since the positions of particles with

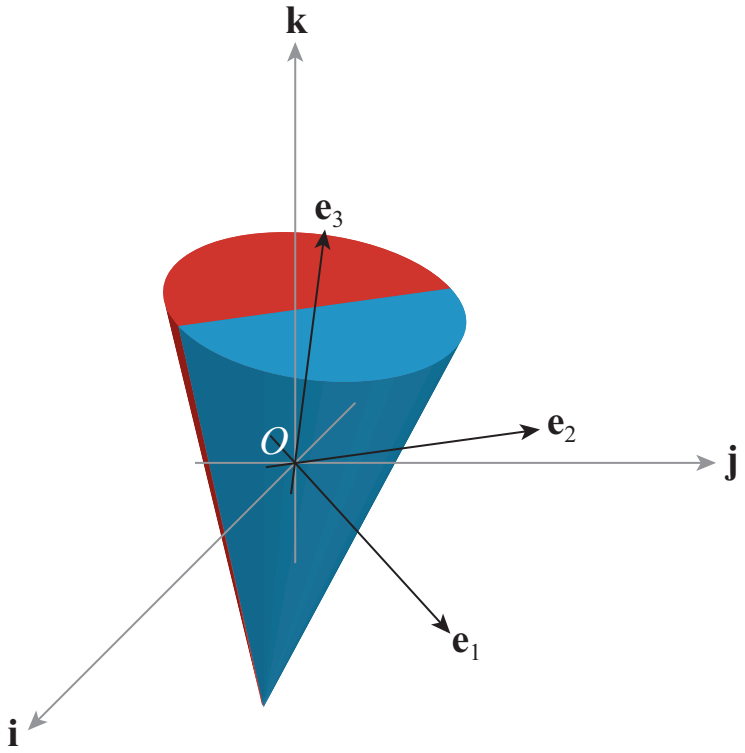

Figure 9.1: Angular velocity and angular momentum representing a general rotation of a dumbell.

respect to the body coordinate are fixed, we can track the motion of the rigid body by following how the body coordinate moves in time. In other words, rigid body kinematics is a matter of identifying the orientation of the body coordinate with respect to the fixed coordinate.

From here on, we will denote the unit vectors for three Cartesian components of the fixed coordinate and the body coordinate with $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, respectively.

9.5 Principal Axes of Inertia

Things are considerably simpler to describe rotation in two dimensions. Both angular velocity and angular momentum are in the same direction, perpendicular to the two dimensional surface and the moment of inertia is always scalar. In three dimensional space, this is generally not the case. Only when certain symmetry conditions are met, rotation of a rigid body can be described in a manner similar to that of two dimensions.

Imagine a dumbbell connected by masses m_1 and m_2 at the ends of its shaft as shown in Fig. 9.5. In general, the angular velocity $\boldsymbol{\omega}$ and the angular momentum $\mathbf{L}$ are not collinear, *i.e.* they are not along the same direction. In other words, $\boldsymbol{\omega}$ and $\mathbf{L}$ are not connected by a scalar number. There are however, two distinct cases in which $\boldsymbol{\omega}$ and $\mathbf{L}$ are collinear. One case is when the rotation

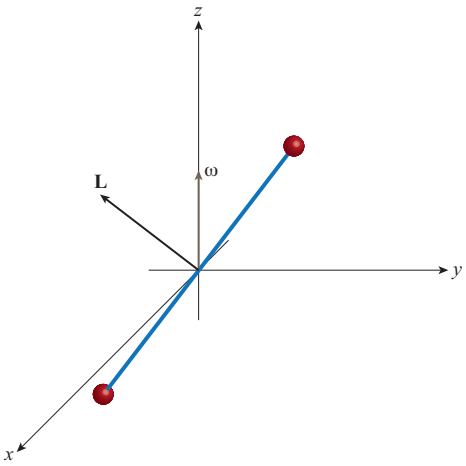

Figure 9.2: Angular velocity and angular momentum representing a general rotation of a dumbell.

axis is along the shaft of the dumbbell, and the other is when the rotation axis is perpendicular to the dumbbell. These particular set of axes are called the principal axes of inertia. In general physics, rotation was always thought to be taking place along these axes. Thus, the relation between the angular velocity and angular momentum could be expressed as

$$\mathbf{L} = I\boldsymbol{\omega} \tag{9.20}$$

with I as a scalar number representing the moment of inertia for the rotation along the principal axis.

The relation Eq. (9.20) can be used to find the principal moment of axes for a rigid body within the given coordinate system. When a body is rotating about a principal axis, we set the angular momentum vector $\mathbf{L} = I\boldsymbol{\omega}$ to be equal to $\mathbf{L} = \mathbb{I} \cdot \boldsymbol{\omega}$.

$$\begin{cases} L_x = I\omega_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y = I\omega_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z = I\omega_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{cases} \tag{9.21}$$

We can rewrite the equations as follows

$$\begin{cases} (I_{xx} - I)\omega_x + I_{xy}\omega_y + I_{xz}\omega_z = 0 \\ I_{yx}\omega_x + (I_{yy} - I)\omega_y + I_{yz}\omega_z = 0 \\ I_{zx}\omega_x + I_{zy}\omega_y + (I_{zz} - I)\omega_z = 0 \end{cases} \tag{9.22}$$

For ω_αs to have non-trivial solutions, the determinant of the coefficients has to be zero.

$$\begin{vmatrix} I_{xx} - I & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} - I & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} - I \end{vmatrix} = 0 \tag{9.23}$$

The equation is third order equation with respect to I and yields three values of I: A, B and C. For each value of $I^\chi = A, B, C$ ($\chi = A, B, C$), we can find

the ratio between ω_x^χ, ω_y^χ and ω_z^χ by solving the equation

$$\begin{cases} (I_{xx} - I^\chi)\omega_x^\chi + I_{xy}\omega_y^\chi + I_{xz}\omega_z^\chi = 0 \\ I_{yx}\omega_x^\chi + (I_{yy} - I^\chi)\omega_y^\chi + I_{yz}\omega_z^\chi = 0 \\ I_{zx}\omega_x^\chi + I_{zy}\omega_y^\chi + (I_{zz} - I)\omega_z^\chi = 0 \end{cases} \tag{9.24}$$

This ratio defines the direction of $\boldsymbol{\omega}^\chi$ along which the moment of inertia has the scalar value I^χ. In other words, the direction defined by $\boldsymbol{\omega}^\chi$s are the principal axes of inertia. In linear algebraic terms, this is the eigenvalue and eigenvector problem, where the principal axes represented by ω^χ are eigenvalues associated to the principal moment of inertia, the eigenvectors.

Using the fact that the moment of inertia tensor is symmetric one can show that the principal axes are orthogonal to one another, that is, $\boldsymbol{\omega}^\chi \cdot \boldsymbol{\omega}^\phi = 0$ for $\chi \neq \phi$. Strictly speaking, if two or three of I^χs have the same value, $\boldsymbol{\omega}^\chi$s need not be orthogonal to each other. Nonetheless, one can always find orthogonal set of $\boldsymbol{\omega}^\chi$s that satisfies Eq. (9.24). We will leave it to the readers to prove this point. (The proof falls into category of linear algebraic problem than a classical mechanics problem.)

Because we now have three orthogonal axes as our princial axes, we can use them as our bases of a coordinate system. In other words, we can represent any vector as a linear combination of three vectors along $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ where $\mathbf{e}_1 \parallel \boldsymbol{\omega}^A$, $\mathbf{e}_2 \parallel \boldsymbol{\omega}^B$ and $\mathbf{e}_3 \parallel \boldsymbol{\omega}^C$.

An important feature of the principal axes is that if we choose them as our basis for the coordinate system, the moment of inertia tensor becomes diagonal, that is

$$\mathbb{I} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \tag{9.25}$$

where the subscripts of I_{ij}s come from the indices of $\mathbf{e}_i$. Therefore the rotation presented within this coordinate system can be expressed through the following relations

$$T = \frac{1}{2}\sum_i I_{ii}\omega_i^2 \tag{9.26}$$

$$L_i = I_{ii}\omega_i \tag{9.27}$$

which are considerably simpler than Eq. (9.18) and (9.12). Due to this simplicity, the principal axes of inertia is an excellent choice as the body coordinate.

9.6 Moments of Inertia for Different Body Coordinate Systems

There are cases when the moment of inertia tensor is given in a particular set of body coordinate system, *e.g.* with the origin of the coordinate system at the center of mass of a rigid body. However, if the body is rotating with the pivot point of rotation not at the center of mass, it is helpful to understand how inertia tensor transforms from the center-of-mass coordinate system to a different body coordinate system. Let's denote three orthogonal axes of the center-of-mass coordinate system with r_is. If we pick another body coordinate system with a

set of axes R_is that has same orientation as the center-of-mass coordinate and is separated by distance a, the translational displacement between the two body coordinate systems is then $\mathbf{a}$, *i.e.* $\mathbf{R} = \mathbf{a} + \mathbf{r}$.

If we substitute each component R_i with $a_i + x_i$, for inertia tensor in the second body coordinate system, J_{ij}, we get

$$\begin{aligned} J_{ij} &= \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha,i} X_{\alpha,j} \right) && (9.28) \\ &= I_{ij} + M(a^2 \delta_{ij} - a_i a_j) && (9.29) \end{aligned}$$

where I_{ij} is the original moment of inertia tensor defined at the center-of-mass body coordinate system and M is the total mass of the rigid body. We will leave the detailed calculation as an exercise for the readers.

The second term in the above equation, $M(a^2\delta_{ij} - a_i a_j)$ is simply moment of inertia of a point particle sitting at a poition $\mathbf{a}$ with respect to the origin. In other words, the moment of inertia tensor in a different coordinate system is the sum of original moment of inertia tensor and the moment of inertia due to a particle with mass M. It can be also shown from this that the orientation of the body coordinates are not affected if the displacement vector $\mathbf{a}$ along one of the three principal axes.

9.7 Euler's Equations

Now that we have established the methods for representing rotational dynamics, we will set up equations of motion:

$$\begin{aligned} \mathbf{F} &= \frac{d\mathbf{P}}{dt} && (9.30) \\ \mathbf{K}_G &= \frac{d\mathbf{L}_G}{dt} && (9.31) \end{aligned}$$

where the first equation is for the translational motion of the center of mass, and the second equation for the rotational motion about the center of mass. If the rigid body has a fixed pivot O different from the center of mass, the equation of motion can be reduced to a single torque equation

$$\mathbf{K}_O = \frac{d\mathbf{L}_O}{dt} \tag{9.32}$$

We will rewrite this equation within the body coordinate system. With $\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$, the angular momentum is expressed as

$$\mathbf{L} = A\omega_1 \mathbf{e}_1 + B\omega_2 \mathbf{e}_2 + C\omega_3 \mathbf{e}_3 \tag{9.33}$$

and the equation of motion is then

$$\begin{aligned} \mathbf{K} &= \frac{d}{dt}\left(I^A \omega_1 \mathbf{e}_1 + I^B \omega_2 \mathbf{e}_2 + I^C \omega_3 \mathbf{e}_3 \right) \\ &= A\dot{\omega}_1 \mathbf{e}_1 + B\dot{\omega}_2 \mathbf{e}_2 + C\dot{\omega}_3 \mathbf{e}_3 \\ &\quad + A\omega_1 \dot{\mathbf{e}}_1 + B\omega_2 \dot{\mathbf{e}}_2 + C\omega_3 \dot{\mathbf{e}}_3 && (9.34) \end{aligned}$$

Note that the coordinate axes are locked to the rigid body, and thus the moment of inertia tensor is unaltered within this coordinate system. Instead, the coordinate axes $\mathbf{e}_i$s are time dependent. Because the coordinate axes are changing with time due to the rotation of the rigid body, the following relation holds, from Eq. (9.5),

$$\dot{\mathbf{e}}_i = \boldsymbol{\omega} \times \mathbf{e}_i \tag{9.35}$$

By plugging $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$ into the above equation, Eq. (9.34) is reduced to

$$\begin{aligned} K_1 &= A\dot{\omega}_1 - (B-C)\omega_2\omega_3 \\ K_2 &= B\dot{\omega}_2 - (C-A)\omega_3\omega_1 \\ K_3 &= C\dot{\omega}_3 - (A-B)\omega_1\omega_2 \end{aligned} \tag{9.36}$$

which is known as the Euler's equations. The equations look quite attractive due to their symmetry, but all the vectors including the torque are decomposed in the body coordinate. Therefore, unless the torque can be easily identified in the body coordinate, the equations are actually not quite useful.

9.8 Free Body Rotation of a Rigid Body with Axial Symmetry

A general motion of a rigid body is extremely complicated to solve, but there are a few cases where the solution comes out in a rather straightforward fashion. One of such problems is the rotation of an axisymmetric body with respect to the center-of-mass in the absence of external torque. Here an axisymmetric body (or a body with axial symmetry) means a body with a cylindrical symmetry, so that $A = B$.

The angular velocity, $\boldsymbol{\omega}$, expressed in terms of the axial unit vector, $\mathbf{e}_3$, is

$$\boldsymbol{\omega} = \omega_3\mathbf{e}_3 + \mathbf{e}_3 \times \dot{\mathbf{e}}_3 \tag{9.37}$$

The first term represents the angular velocity component along the symmetric axis, $\mathbf{e}_3$ and the second term is the angular velocity component perpendicular to the symmetric axis. Thus the angular momentum can be written as

$$\mathbf{L} = A\mathbf{e}_3 \times \dot{\mathbf{e}}_3 + C\omega_3\mathbf{e}_3 = \text{constant} \tag{9.38}$$

where the first term comes from the axial symmetry. Because the system has cylindrical symmetry, we can choose any axis that is perpendicular to $\mathbf{e}_3$ as a principal axis, and the moment of inertia along such axis is equal to A. Since there is no external torque acting on the system, the angular momentum $\mathbf{L}$ must be conserved.

Because the angular momentum is conserved, the following relation is also true:

$$\mathbf{e}_3 \cdot \frac{d\mathbf{L}}{dt} = C\dot{\omega}_3 = 0 \tag{9.39}$$

which implies that the axial component of the angular velocity, ω_3 does not change with time, even though the symmetric axis itself is not constant.

Taking the dot product between $\mathbf{e}_3$ and $\mathbf{L}$ results in

$$\mathbf{e}_3 \cdot \mathbf{L} = C\omega_3 = \text{constant} \tag{9.40}$$

Thus the angle between the angular momentum and the symmetric axis is also constant in time.

This time, we will take the cross product between $\mathbf{e}_3$ and $\mathbf{L}$ and get

$$\begin{aligned} \mathbf{e}_3 \times \mathbf{L} &= A\mathbf{e}_3 \times (\mathbf{e}_3 \times \dot{\mathbf{e}}_3) + C\omega_3 \mathbf{e}_3 \times \mathbf{e}_3 \\ &= -A\dot{\mathbf{e}}_3 \end{aligned} \tag{9.41}$$

By rearranging the above equation, the following can be identified

$$\dot{\mathbf{e}}_3 = \frac{\mathbf{L}}{A} \times \mathbf{e}_3 = \boldsymbol{\Omega} \times \mathbf{e}_3 \tag{9.42}$$

where $\boldsymbol{\Omega} \equiv \mathbf{L}/A$. This equation states that the symmetry axis $\mathbf{e}_3$ precesses around the angular momentum axis $\mathbf{L}$ with constant angular speed L/A.

We will rewrite the angular velocity as

$$\begin{aligned} \boldsymbol{\omega} &= \mathbf{e}_3 \times \dot{\mathbf{e}}_3 + \omega_3 \mathbf{e}_3 \\ &= \mathbf{e}_3 \times \left(\frac{\mathbf{L}}{A} \times \mathbf{e}_3 \right) + \frac{\mathbf{e}_3 \cdot \mathbf{L}}{C} \mathbf{e}_3 \\ &= \frac{\mathbf{L}}{A} + \mathbf{e}_3 \cdot \mathbf{L} \left(\frac{A - C}{AC} \right) \mathbf{e}_3 \end{aligned} \tag{9.43}$$

This can be decomposed as angular velocity of the symmetric axis $\mathbf{e}_3$ about the fixed angular momentum axis $\mathbf{L}$ and the angular velocity of the rigid body about the symmetric axis $\mathbf{e}_3$. In other words, the rigid body rotates about $\mathbf{e}_3$ with an angular speed $\mathbf{e}_3 \cdot \mathbf{L} \left(\frac{A-C}{AC} \right)$ and the rotating body precesses about the fixed axis with an angular speed L/A.

The same result can be derived by solving the Euler's equations from the previous section. One can also show that three vectors $\mathbf{L}$, $\boldsymbol{\omega}_3$ and $\mathbf{e}_3$ are all on the same plane, which is a good vector calculus exercise for the readers.

9.9 Precession of a Symmetric Top due to a Weak Torque

Consider a symmetric top whose inertia tensor takes the form

$$\mathbb{I} = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix} \tag{9.44}$$

relative to the top's principal axes, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. Here $\mathbf{e}_3$ is the symmetry axis, and $\mathbf{e}_1$ and $\mathbf{e}_2$ can be chosen to be any two orthogonal axes perpendicular to $\mathbf{e}_3$. See Fig. 9.9.

If the rotation of the body is about the axis of symmetry, *i.e.* $\boldsymbol{\omega} = \omega_3 \mathbf{e}_3$, the angular momentum is simply,

$$\mathbf{L} = C\boldsymbol{\omega} = C\omega_3 \mathbf{e}_3 \tag{9.45}$$

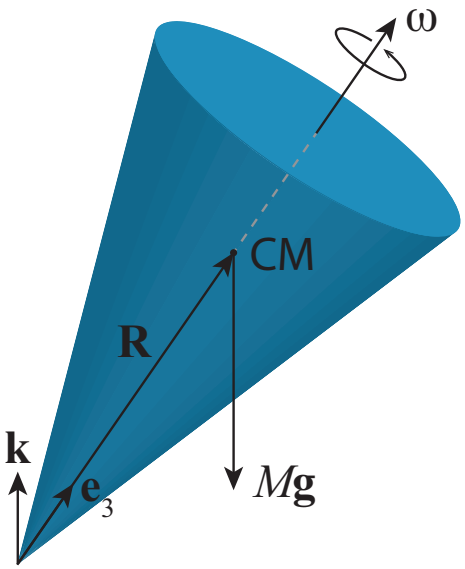

Figure 9.3: A weak torque acting on a spinning top.

Torque due to gravity for such a top is

$$\boldsymbol{\Gamma} = \mathbf{R} \times M\mathbf{g} = \dot{\mathbf{L}} = C\omega_3 \dot{\mathbf{e}}_3 \tag{9.46}$$

where $\mathbf{R}$ is the center of mass position with respect to the fixed pivot point O and $\mathbf{g}$ is the gravity along the negative z-axis, *i.e.* $\mathbf{g} = -g\hat{\mathbf{k}}$. The total mass of the top is given by M. We can solve for $\dot{\mathbf{e}}_3$,

$$\dot{\mathbf{e}}_3 = \frac{MgR}{C\omega_3}\hat{\mathbf{k}} \times \mathbf{e}_3 = \boldsymbol{\Omega} \times \mathbf{e}_3 \tag{9.47}$$

where

$$\boldsymbol{\Omega} = \frac{MgR}{C\omega_3}\hat{\mathbf{k}} \tag{9.48}$$

This again implies that the spinning top will precess with the angular speed $\Omega = \frac{MgR}{C\omega_3}$. An implicit condition here is that the precession frequency Ω has to be substantially smaller than ω_3. Otherwise, the angular velocity $\boldsymbol{\Omega}$ of the precession alters the total angular velocity of the spinning top sufficiently that the angular velocity will no longer be along the symmetry axis $\mathbf{e}_3$. This violates the initial assumption of our problem.

9.10 Steady Precession of a Symmetric Top under a Uniform Torque

From what we have discussed up to this point, we can find the general condition for a steady precession of a spinning top under gravity. One can easily derive the following equation of motion

$$\frac{d}{dt}\left(A\mathbf{e}_3 \times \dot{\mathbf{e}}_3 + C\omega_3\mathbf{e}_3\right) = -MgR(\mathbf{e}_3 \times \mathbf{k}) \tag{9.49}$$

Steady precession is a precession of a top in which the angle, α, between the axial vector $\mathbf{e}_3$ and the vertical $\mathbf{k}$ and the precession frequency are fixed, *i.e.* $\mathbf{e}_3 \cdot \mathbf{k} = \cos\alpha =$ constant and $\Omega =$ constant. In such steady precession, the rate of change of $\mathbf{e}_3$ is given by

$$\dot{\mathbf{e}}_3 = \Omega\mathbf{k} \times \mathbf{e}_3 \tag{9.50}$$

If such motion is possible, one should be able to rewrite the first term in the Eq. (9.49) as

$$\begin{aligned} \frac{d}{dt}\left(A\mathbf{e}_3 \times \dot{\mathbf{e}}_3\right) &= A\frac{d}{dt}\left\{\mathbf{e}_3 \times (\Omega\mathbf{k} \times \mathbf{e}_3)\right\} \\ &= A\Omega\frac{d}{dt}\left(\mathbf{k} - \cos\alpha\,\mathbf{e}_3\right) \\ &= A\Omega^2\cos\alpha\,(\mathbf{e}_3 \times \mathbf{k}) \end{aligned} \tag{9.51}$$

and the second term as

$$\frac{d}{dt}(C\omega_3\mathbf{e}_3) = C\omega_3\dot{\mathbf{e}}_3 = -C\omega_3\Omega(\mathbf{e}_3 \times \mathbf{k}) \tag{9.52}$$

Using the relations given by Eq. (9.51) and (9.52), Eq. (9.49) can be simplified to

$$(A\cos\alpha\Omega^2 - C\omega_3\Omega + MgR)(\mathbf{e}_3 \times \mathbf{k}) = 0 \tag{9.53}$$

For the precession frequency Ω to have a real value, the following condition has to be met

$$(C\omega_3)^2 \geq 4AMgR\cos\alpha \tag{9.54}$$

One can easily show that for the case $(C\omega_3)^2 \gg 4AMgR\cos\alpha$, two solutions to the quadratic equation for Ω,

$$A\cos\alpha\Omega^2 - C\omega_3\Omega + MgR = 0 \tag{9.55}$$

becomes the two different precession frequencies obtained in the previous two sections:

$$\Omega^F \approx \frac{L}{A} \qquad \text{and} \qquad \Omega^S \approx \frac{MgR}{C\omega_3} \tag{9.56}$$

Readers can ponder upon the meaning of this statement.

9.11 Eulerian Angles

So far, we have learned how to describe rotation of rigid body in a few special cases. Directly solving for the equation of motion or a rigid body is, in most cases, not at all straightforward. Instead, one can utilize Lagrangian mechanics which requires an appropriate choice of generalized coordinates. One widely used generalized coordinates are the Eulerian angles.

To understand what Eulerian angles are, we have to rethink about the relation between the fixed coordinate system and the body coordinate system which is geometrically locked to the rigid body of interest. Since both coordinate systems are orthogonal three dimensional coordinate systems, there must be a coordinate transformation that transforms the fixed coordinates into the body coordinates and vice versa. If the two coordinate systems share a common origin, the coordinate transformation is a set of rotations.

There is always a single rotation about a particular axis that will transform the fixed coordinates into the body coordinates. However, this is not the simplest method as it requires four parameters: three parameters to identify the axis of rotation and the forth parameter to identify the amount of rotation, *i.e.* the angle of rotation. Instead, a series of three rotations about known axes can do the job just as well. There are different choices for such series of rotations, but we will follow the method developed by Euler.

Consider the following series of rotations, which takes the x_i' system into the x_i system.

1. Rotation by ϕ counterclock-wise about x_3'-axis: the transformation leads to
$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} \tag{9.57}$$
Because the rotation took place about the x_3'-axis, it is unaffected by the rotation and $x_3'' = x_3'$.

2. Rotation by θ counterclock-wise about x_1''-axis:
$$\begin{pmatrix} x_1''' \\ x_2''' \\ x_3''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} \tag{9.58}$$
This rotation will lead to the x_i''' system with $x_1'' = x_1'''$.

3. Finally, rotation by ψ counterclock-wise about $x_3'''(= x_3)$-axis:
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1''' \\ x_2''' \\ x_3''' \end{pmatrix} \tag{9.59}$$

The final coordinate system is the x_i system with $x_3 = x_3'''$. Since these three subsequent rotations will result in coordinate transformation from x_i system to x_i, we can write the relation as

$$\begin{aligned} \mathbf{x} &= \boldsymbol{\lambda}\mathbf{x}' \text{ where} \\ \boldsymbol{\lambda} &= \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \end{aligned}$$

As difficult as it is for one to see it, the beauty of this is that one can transform between any two Cartesian coordinate systems that share a common origin using this method with three parameters ϕ, θ and ψ and these three angles are known as the Eulerian angles.

Our goal was to find appropriate coordinate transformation that yields $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ system from **ijk** system. In this case **k**- and $\mathbf{e}_3$-axes play the role of x_3' and x_3-axes, respectively and θ is the angle between **k** and $\mathbf{e}_3$, *i.e.* $\cos\theta = \mathbf{e}_3 \cdot \mathbf{k}$. We still need to identify x_1''-axis, which turns out to be the straight line formed by the **ij**-plane and $\mathbf{e}_1\mathbf{e}_2$-plane. This is equivalent to the axis perpendicular to the plane formed by $\hat{\mathbf{k}}$ and $\mathbf{e}_3$ vectors.

Therefore, one can get to $\mathbf{e}_i$ system from the **ijk** system by 1) rotating the **ijk** coordinate system about the **k**-axis until the **i**-axis matches $\mathbf{i}' = \mathbf{k} \times \mathbf{e}_3$. We will denote the amount of rotated angle by ϕ: $\phi = \cos^{-1}\{\mathbf{i} \cdot (\mathbf{k} \times \mathbf{e}_3)\}$. 2) Rotate the previously achieved coordinate system about $\mathbf{i}'$ by $\theta = \cos^{-1}(\mathbf{e}_3 \cdot \mathbf{k})$. 3) Finally, rotate the coordinate system about $\mathbf{e}_3$ until the coordinate axes are all aligned to $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ system, that is $\mathbf{i}'$ becomes $\mathbf{e}_1$. The final angle of rotation is denoted by $\psi = \cos^{-1}\{(\mathbf{k} \times \mathbf{e}_3) \cdot \mathbf{e}_1\}$. Note that all the angles can be defined by the basis vectors of the two coordinate systems between which the transformation is occurring.

9.12 Motion of a Symmetric Top with One Point Fixed

If $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ system represents the body coordinates of a rigid body under rotation, then the basis vectors for this coordinate system $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ must change with time. However, this is not a problem per se, since any time dependence of the coordinate axes can be dumped into the Eulerian angles. Therefore, the rigid body dynamics simply becomes the matter of finding time dependence of $\phi(t)$, $\theta(t)$ and $\psi(t)$ and they can be the perfect choice as the generalized coordinates.

This is especially useful considering how ineffective Euler's equations (Eq. (9.36)) are. The problem is that the equations themselves are set up in the body coordinate system, and unless the solution is known or guessed correctly in advance, there is no good way to identify the external torque of the system.

To overcome this shortcoming, we will introduce Lagrangian mechanics. Let us first identify angular velocity in terms of Eulerian angles. Angular velocity $\boldsymbol{\omega}$ can be constructed with vectors having $\hat{\mathbf{z}}$, $\hat{\mathbf{x}}'$ and $\mathbf{e}_3$ as basis, $\boldsymbol{\omega} = \dot{\boldsymbol{\phi}} + \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{\psi}}$. We will denote

$$\begin{cases} \dot{\phi} &= \text{angular speed about } \hat{\mathbf{z}} \\ \dot{\theta} &= \text{angular speed about } \hat{\mathbf{x}}' = \hat{\mathbf{z}} \times \mathbf{e}_3 \\ \dot{\psi} &= \text{angular speed about } \mathbf{e}_3 \end{cases} \tag{9.60}$$

We can consider projection of each of these angular velocities along three body coordinate axes, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

$$\begin{cases} \dot{\phi}_1 &= \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}_2 &= \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}_3 &= \dot{\phi}\cos\theta \end{cases} \tag{9.61}$$

$$\begin{cases} \dot{\theta}_1 &= \dot{\theta}\cos\psi \\ \dot{\theta}_2 &= \dot{\theta}\sin\psi \\ \dot{\theta}_3 &= 0 \end{cases} \tag{9.62}$$

$$\begin{cases} \dot{\psi}_1 &= 0 \\ \dot{\psi}_2 &= 0 \\ \dot{\psi}_3 &= \dot{\psi} \end{cases} \tag{9.63}$$

From this, we can write the kinetic energy of the system

$$\begin{aligned} T &= \frac{1}{2}\sum I_{ii}\omega_i^2 && (9.64)\\ &= \frac{1}{2}A(\omega_1^2+\omega_2^2)+\frac{1}{2}C\omega_3^2 && (9.65)\\ &= \frac{1}{2}A(\dot\phi^2\sin^2\theta+\dot\theta^2)+\frac{1}{2}C(\dot\phi\cos\theta+\dot\psi)^2 && (9.66)\end{aligned}$$

If the center of mass of the rigid body is distance R away from the rotating center in the body coordinate system, the potential energy of the system is given by

$$V = MgR\cos\theta \tag{9.67}$$

The Lagrangian $L = T - V$ is cyclical in ϕ and ψ, therefore

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial\dot\phi} = (A\sin^2\theta + C\cos^2\theta)\dot\phi + C\dot\psi\cos\theta = \text{constant} && (9.68)\\ p_\psi &= \frac{\partial L}{\partial\dot\psi} = C(\dot\psi+\dot\phi\cos\theta) = \text{constant} && (9.69)\end{aligned}$$

Eq. (9.69) yields

$$\dot\psi = \frac{p_\psi - C\dot\phi\cos\theta}{C} \tag{9.70}$$

and Eq. (9.70) combined with Eq. (9.68) gives us

$$\dot\phi = \frac{p_\phi - p_\psi\cos\theta}{A\sin^2\theta} \tag{9.71}$$

We can then rewrite Eq. (9.70) as

$$\dot\psi = \frac{p_\psi}{C} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{A\sin^2\theta} \tag{9.72}$$

Note that the angular speeds $\dot\phi$ and $\dot\psi$ are now expressed in terms of a single variable θ.

We now write the total energy of the system

$$E = \frac{1}{2}A(\omega_1^2+\omega_2^2)+\frac{1}{2}C\omega_3^2 + MgR\cos\theta \tag{9.73}$$

which must be constant. Since $p_\psi = C\omega_3$ is constant, $\frac{1}{2}C\omega_3^2$ must also be constant. Therefore we can define a different energy scale $E' \equiv E - \frac{1}{2}C\omega_3^2$ which is also constant.

$$\begin{aligned} E' &= \frac{1}{2}A(\omega_1^2+\omega_2^2)+MgR\cos\theta && (9.74)\\ &= \frac{1}{2}A(\dot\phi^2\sin^2\theta+\dot\theta^2)+MgR\cos\theta && (9.75)\\ &= \frac{1}{2}A\dot\theta^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2A\sin^2\theta}+MgR\cos\theta && (9.76)\\ &= \frac{1}{2}A\dot\theta^2 + V(\theta) && (9.77)\end{aligned}$$

where $V(\theta) \equiv \frac{(p_\phi - p_\psi \cos\theta)^2}{2A\sin^2\theta} + MgR\cos\theta$ is the effective potential energy in θ-space. One can easily see that Eq. (9.77) is a one dimensional problem in θ-space with rotational kinetic energy $\frac{1}{2}A\dot{\theta}^2$ and effective potential energy $V(\theta)$. From this equation, we can calculate $t(\theta)$

$$t(\theta) = \int \frac{d\theta}{\sqrt{(2/A)[E' - V(\theta)]}} \tag{9.78}$$

and the final result can be inverted to obtain $\theta(t)$. Because $\dot{\phi}$ and $\dot{\psi}$ were expressed as functions of θ only in Eq. (9.71) and (9.72), $\theta(t)$ can be used to find $\phi(t)$ and $\psi(t)$. Without actually going into the full solution, which cannot be analytically solved anyways, we can deduce qualitative motion by looking into $V(\theta)$ in Eq. (9.77). The effective potential is given by

$$V(\theta) \equiv \frac{(p_\phi - p_\psi \cos\theta)^2}{2A\sin^2\theta} + MgR\cos\theta \tag{9.79}$$

and one can easily see that the potential diverges at $\theta = 0$ and π due to the $\sin^2\theta$ in the denominator of the first term. If the total energy of the system happens to coincide with the minimum of $V(\theta)$, the spinning top will precess steadily with $\theta = \theta_0$. Here θ_0 satisfies the condition $V(\theta_0) = V_{\min}$. One can find the precession speed by calculating $\dot{\phi}$ for $\theta = \theta_0$. The result should be identical to the solutions to the Eq. (9.55). This is a good exercise for readers.

In general, the angle θ will have a bound motion between some angles θ_1 and θ_2. One can easily show that the angular velocity about the z-axis is non-zero from Eq. (9.71). Therefore, the general motion of a spinning top includes an oscillation of the top's symmetry axis between θ_1 and θ_2 superposed to the precession around the z-axis. Such oscillatory precession is usually referred to as nutation. Eq. (9.71) can be used to see if $\dot{\phi}$ changes sign at any point, meaning whether the nutation includes a looping motion.